Mathematics Support Capsules

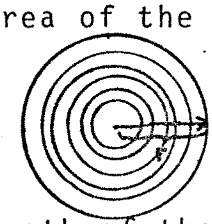
COMPOSITE FUNCTIONS

by King K. Chan

There is a method for combining two functions by substituting one function into another. The result of this combination is a composite function.

The following is an example* of a combination leading to a composite function. The example also shows how composite functions are used in the real world. (the real school world)

You throw a rock in a lake. As the rock hits the water, water is displaced forming rings around the spot where the rock hit. You want to know the area of the rings (circles) at any given time



$$\text{Area} = \pi\, r^2 \qquad (1)$$

But the length of the radius (r) will increase as time as time increases. (In fact, you found that the radius increases 10 ft. per minute.) This relationship can be written as

$$r = 10\, t \qquad (2)$$

Substituting equation (2) into equation (1), we get the formula for Area of the rings at any time as:

$$A = \pi(10\, t)^2 \quad = \quad \boxed{A = \pi\, 100\, t^2}$$

The last function is a composite function, which is really the function of a function. A is a function of r , r is a function of t.

The remainder of this module provides systematic methods to solve for a composite function and its range and domain.

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* paraphrased from Hughes-Hallett - see Reference.

Solving for the Composite Function $g(h(x))$ (or $g \circ h$).

We can find the composition of g with h by substituting for x-values of the first function $(y_1 = g(x))$, $h(x)$ from the second function $(y_2 = h(x))$

$$g(x) = ax^2 + bx \qquad h(x) = cx$$

$$g(h(x)) = a(cx)^2 + b(cx)$$

Given 2 functions:

$$g(x) = \frac{1}{3-x} \qquad h(x) = \sqrt{x} + 1$$

The procedure for solving for the composite function is

1) Establish the functions in their simplest form

$$g(x) = \frac{1}{3-x}$$

$$h(x) = \sqrt{x} + 1$$

2) Substitute for x of the first function, h(x) from the second function.

$$\frac{1}{3 - (h(x))} = \frac{1}{3 - \sqrt{x} + 1)}$$

3) Simplify the resulting composite function.

$$g(h(x)) = \frac{1}{3 - (\sqrt{x} + 1)}$$

$$= \frac{1}{2 - \sqrt{x}}$$

Solving for the composite function: $h(g(x))$.

In this case, the composition of g with h is reversed where $h(x)$ is given as a function of $g(x)$. The procedure to solve for the composite function $Y = h(g(x))$ is the same as the procedure to solve for $Y = g(h(x))$ except we now substitute for x in the second equation with $g(x)$ from the first equation.

Try it yourself. The answer should be:

$$h(g(x)) = \sqrt{\left(\frac{1}{3-x}\right)} + 1.$$

NOTE: As you can see, $g(h(x)) \neq h(g(x))$. Therefore, you should make certain you understand which composition you are asked to perform.

EXERCISES:

1) Find $g(h(x))$ and $h(g(x))$. $\quad g(x) = x^2 + 5 \quad h(x) = 2x + 1$

2) Find $g(h(x))$ and $h(g(x))$. $\quad g(x) = \frac{x - 1}{x + 1} \quad h(x) = \frac{x + 1}{x - 1}$

3) Find $g(g(x))$. $\quad g(x) = x^2 - 9$

4) $g(x) = x^2 + 1 \quad h(x) = \frac{1}{x + 1} \quad$ Find: $g\left(\frac{1}{h(x)}\right)$ and $\frac{1}{g(h(x))}$.

Answers on following page.

Answers to exercises:

1)
$$g(h(x)) = (2x+1)^2+5$$
$$\text{or } 4x^2+4x+6$$

$$h(g(x)) = 2(x^2+5)+1$$
$$= 2x^2+11$$

2)
$$g(h(x)) = \frac{\left(\frac{x+1}{x-1}\right)-1}{\left(\frac{x+1}{x-1}\right)+1}$$
$$= \frac{\frac{x+1}{x-1} - \frac{x-1}{x-1}}{\frac{x+1}{x-1} + \frac{x-1}{x-1}}$$
$$= \frac{x+1-(x-1)}{x+1+x-1}$$
$$= \frac{2}{2x}$$
$$= \frac{1}{x}$$

$$h(g(x)) = \frac{\left(\frac{x-1}{x+1}\right)+1}{\left(\frac{x-1}{x+1}\right)-1}$$
$$= \frac{\frac{x-1}{x+1} + \frac{x+1}{x+1}}{\frac{x-1}{x+1} - \frac{x+1}{x+1}}$$
$$= \frac{x-1+x+1}{x-1-(x+1)}$$
$$= \frac{2x}{-2} = -x$$

3)
$$g(g(x)) = (x^2-9)^2-9$$
$$\text{or } (x^4-18x^2+81)-9$$
$$= x^4-18x^2+72$$

4)
$$\frac{1}{h(x)} = \frac{1}{\frac{1}{x+1}} = x+1$$
$$f\left(\frac{1}{h(x)}\right) = (x+1)^2+1$$
$$\text{or } (x^2+2x+1)+1$$
$$= x^2+2x+2$$

$$f(h(x)) = \left(\frac{1}{x+1}\right)^2+1$$
$$= \frac{1}{x^2+2x+1}+1$$
$$= \frac{x^2+2x+2}{x^2+2x+1}$$
$$\frac{1}{f(h(x))} = \frac{x^2+2x+1}{x^2+2x+2}$$

Finding the domain of the composite function, $g(h(x))$.

Def: A domain of a function $y = f(x)$ is the set of possible x-values of that function.

According to Thomas (4th ed.) the domain of a composite function is made up of those x-values of the second function whose corresponding $h(x)$ (y-values) is among the possible x-values of the first function.

The procedure for finding the domain of $g(h(x))$:

1) Determine $g(x)$ and $h(x)$ in their simplest form and the composite function $g(h(x))$.

$g(x) = \frac{1}{3 - x}$

$h(x) = \sqrt{x} + 1$

$g(h(x)) = \frac{1}{2 - \sqrt{x}}$

2) Determine the possible x values of the second function, $h(x)$.

$h(x) = \sqrt{x} + 1$

$x \geq 0$, ← the square root of a negative number is imaginary.

We are only dealing with real numbers.

3) Out of the possible x values of the second function (in this example $x \geq 0$) determine which $h(x)$'s are among the possible x-values of the first function, $g(x)$.

$g(x) = \frac{1}{3 - x}$

$x \neq 3$ ← cannot divide by 0

This means $h(x)$ can be any number but 3, or $h(x) \neq 3$

since $h(x) = \sqrt{x} + 1$

$\sqrt{x} + 1 \neq 3$

$\sqrt{x} \neq 2$

$x \neq 4$

4) The possible values for x for the composite function $g(h(x))$ is given by the intersection of (2) and (3).

$0 \leq x < \infty \quad x \neq 4$

Find the range for $g(h(x))$.

> Def: A range of a function $y = f(x)$ is the set of possible y-values of that function.

We can now use the domain to solve for the range. We will determine the range of $Y = g(h(x))$ given the domain of this composite function for which we have just solved.

Procedure to solve for the range of $g(h(x))$.

1) We have just solved for the domain from the last section.	$(x \geq 0,\ x \neq 4)$
2) Substitute these x-values into the composite function	$g(h(x)) = \dfrac{1}{2 - \sqrt{0}} = \dfrac{1}{2}$ if $x \geq 0$, $g(h(x)) \geq \frac{1}{2}$ $g(h(x)) = \dfrac{1}{2 - \sqrt{x}}$ is not defined at $x = 4$ because the denominator would then be 0. Here, $x \neq 4$ is a restraint so that $g(h(x))$ will not be undefined.
3) The range is	$x \leq y < \infty$

> When solving for the domain and range, be careful of unusual symbols. (e.g. $\sqrt{x}$, $\frac{1}{x}$, $|x|$, $(x)^2$) $\sqrt{x}$ will limit the domain to positive numbers. $\frac{1}{x}$ will eliminate $x = 0$ from the domain. $|x|$ and $(x)^2$ will limit the range to mostly positive numbers.

Finding the domain and range of the composite function $h(g(x))$.

We have already seen that $g(h(x)) \neq h(g(x))$. We can then conclude that their domains and ranges will most likely also differ.

But, the procedure for solving for the domain and range are the same for both composite functions, except $h(x)$ is now our first function, and $g(x)$ is our second function. Try solving them for yourself, then check the solutions on the next page.

$g(x) = \frac{1}{3-x}$; $h(x) = \sqrt{x} + 1$

$h(g(x)) =$

Solution for h(g(x)) - Domain and Range:

Step	Comment

domain ↓

(1) $h(x) = \sqrt{x} + 1$

$g(x) = \frac{1}{3-x}$

$h(g(x)) = \sqrt{\frac{1}{3-x}} + 1$

(2) $g(x) = \frac{1}{3-x}$, $\underline{x \neq 3}$. (x=3 implies that g(x) is undefined)

(3) $h(x) = \sqrt{x} + 1$, $x \geq 0$ (x < 0 implies that we will have imaginary numbers.)

$\frac{1}{3-x} \geq 0$ (Since the numerator is always positive, the denominator must be positive.)

$3 - x \geq 0$ $\underline{x \leq 3}$

(4) $x \neq 3$ *and* $x \leq 3$ implies that

$\boxed{-\infty < x < 3}$

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range ↓

(1) $-\infty < x < 3$

(2) We don't evaluate h(g(x)) at x = 3 because we've excluded this from our domain.

but at x = 2.9999

$h(g(x)) = \sqrt{\frac{1}{3.-2.9999}} + 1$

$= 101$

at x = 2.999999

$h(g(x)) = \sqrt{\frac{1}{3-2.999999}} + 1 = 1001$

We see as x gets closer to 3, h(g(x)) gets very large. Our upper bound for y is $y < \infty$.

As x gets larger negatively (i.e. $x \to -\infty$)

$\sqrt{\frac{1}{3-(x)}}$ gets very small ($\sqrt{\frac{1}{3-(x)}} \to 0$)

but: $h(g(x) = \sqrt{\frac{1}{3-(x)}} + 1$, so : as x gets larger, $h(g(x))$ gets closer to 1. Our lower bound for y is $y > 1$

(4) The range is $\boxed{1 < y < \infty}$

Exercises:

Given the following functions, find the composite functions $g(h(x))$ and $h(g(x))$; and their respective domains and ranges.

1) $g(x) = 3x - 5$ $h(x) = 2 - x$

2) $g(x) = 2x + 5$ $h(x) = x^2$

3) $g(x) = 2x + 4$ $h(x) = |x|$

4) $g(x) = \sqrt{x^3 - 1}$ $\quad\quad$ $h(x) = 3x - 1$

5) $g(x) = \frac{1}{x}$ $\quad\quad$ $h(x) = \frac{1}{x^2}$

Hint: In (5), what happens to $x = 0$ in $h(x)$ and $g(h(x))$? Go back to the definition of the domain of $g(h(x))$. Same for $x = 0$ in $g(x)$ and $h(g(x))$.

6) Given the following functions, find $f(g(h(x)))$ or $f \circ g \circ h$; and the composite function's domain and range

$$f(x) = \frac{x + 1}{x} \qquad g(x) = \frac{1}{2x + 1} \qquad h(x) = x^2$$

Answers on next page

Answers to exercises.

1) $g(h(x)) = 3(2-x) - 5$
$= 1 - 3x$

D: $-\infty < x < \infty$

R: $-\infty < y < \infty$

$h(g(x)) = 2 - (3x-5)$
$= 7 - 3x$

$-\infty < y < \infty$

$-\infty < y < \infty$

2) $g(h(x)) = 2(x^2) + 5$
$= 2x^2 + 5$

D: $-\infty < x < \infty$

R: $5 \leq y < \infty$

$h(g(x)) = (2x+5)^2$

$-\infty < x < \infty$

$0 \leq y < \infty$

Note: In cases with $(ax+b)^2$, it is better to leave it in this form and not multiply it out. In this case $(2x+5)^2$ tells us that the range cannot be negative because of the square.

3) $g(h(x)) = 2(|x|) + 4$

D: $-\infty < x < \infty$

R: $4 \leq y < \infty$

↑

if $|x| \leq 0$, then 0 is smallest x-value,
$2(0) + 4 = 4$

$h(g(x)) = |2x + 4|$

$-\infty < x < \infty$

$0 < y < \infty$

↑

because $|2x + 4|$ must be greater than 0.

4) $g(h(x)) = \sqrt{(3x-1)^3 - 1}$

$2/3 \leq x \leq \infty$

$0 \leq y < \infty$

$(3x-1)^3 - 1 \geq 0$ because $\sqrt{\ }$ sign. Solve for
$(3x-1)^3 - 1 \geq 0$,
$x \geq 2/3$

$h(g(x)) = 3(\sqrt{x^3-1}) - 1$

$1 \leq x < \infty$

$-1 \leq y < \infty$

$x^3 - 1 \geq 0$ because of $\sqrt{\ }$.
Solve $x^3 - 1 \geq 0$, $x \geq 1$

5) $g(h(x)) = \dfrac{1}{\left(\frac{1}{x^2}\right)} = x^2$ $\qquad$ $h(g(x)) = \dfrac{1}{\left(\frac{1}{x}\right)^2} = x^2$

D: $-\infty < x < \infty \quad x \neq 0$ $\qquad$ $-\infty < x < \infty \quad x \neq 0$

R: $0 < y < \infty$ $\qquad$ $0 < y < \infty$

In both $g(h(x))$ and $h(g(x))$, $x \neq 0$ because the value 0 was restricted from the domains of the original functions. Although 0 may be a solution to the composite functions, it is not a solution to the original functions.

6) $g(h(x)) = \dfrac{1}{2x^2 + 1}$

D: $-\infty < x < \infty$

R: $1 \leq y < \infty$

In order to find $f(g(h(x)))$ and its domain and range, we must solve $g(h(x))$ and its domain and range.

$$f(g(h(x))) = \frac{\left(\frac{1}{2x^2+1}\right) + 1}{\frac{1}{2x^2+1}}$$

$$= \frac{\frac{1}{\cancel{2x^2+1}} + \frac{2x^2+1}{\cancel{2x^2+1}}}{\frac{1}{\cancel{2x^2+1}}}$$

$$= 2x^2 + 2$$

D: $-\infty < x < \infty$

R: $2 \leq y < \infty$

For more work on composite functions, see

Deborah Hughes-Hallett, The Math Workshop: Elementary Functions (W. W. Norton, 1980), pp.173-181.

or try any calculus or precalculus text. See also the Mathematics Support Capsule on INVERSE FUNCTIONS.

ese Mathematics Support Capsules were prepared under the supervision of Beverly West, Lecturer, Department of Mathematics, rnell University, Ithaca, New York 14853, with assistance from King Chan, Ken Gardner, Kristen Jackson, Ann Michel, Ken amoto, and funding from the Exxon Corporation. Reproduction of these items for any commercial purpose is expressly ohibited. The authors solicit your comments, corrections, and suggestions for future revision. August, 1981