

Mathematics Support Capsules

EXPONENTIAL AND LOGARITHMIC FUNCTIONS, WITH CALCULUS

by Sue Barton

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I. DEFINITIONS AND PROPERTIES OF LN X

This capsule will cover the natural logarithm and the natural exponential function. It will then go on to give a short discussion on general logarithms and general exponential functions. All of this will be done with the assumption of some knowledge of calculus.

In precalculus we were told to define $\log_a x$ so that

$$a^x = y \iff \log_a y = x$$

Logarithms, as defined, have many useful properties and you may remember some of them from precalculus. You should remember the following rules: (if not, see the precalculus capsule on LOGARITHMS).

1. $\log_n ab = \log_n a + \log_n b$
2. $\log_n a^b = b \cdot \log_n a$
3. $\log_n a/b = \log_n a - \log_n b$

Example: $\log_n \frac{3 \cdot 5}{2^2} = \log_n 3 \cdot 5 - \log_n 2^2$
 $= \log_n 3 + \log_n 5 - 2 \log_n 2$

Now in calculus there is a particular logarithm which is very handy, and we will start by defining it.

Definition: The natural logarithm of x ($\ln x$) is defined as follows:

$$\ln x \equiv \int_1^x \frac{1}{t} dt \quad \text{for all } x > 0$$

Even though this definition may seem obscure we will see it behaves like other logarithms.

Theorem: The following three rules hold for $\ln x$:

1. $\ln ab = \ln a + \ln b$
2. $\ln a^b = b \ln a$
3. $\ln a/b = \ln a - \ln b$

Proof: See any calculus book.

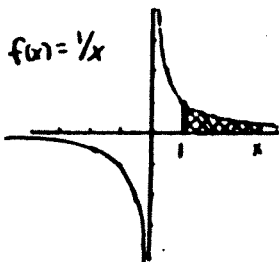
It is important to note that $f(x) = \ln x$ is a function, so just as in general $f(x+3) \neq f(x) + 3$ it is also true that $\ln(x+3) \neq \ln(x) + 3$. To clear up notation it should be noted that

$$\ln(2x+7)^2 \neq \ln(2x+7) \cdot \ln(2x+7)$$

The "square" is on the $2x+7$; not the whole function. Note, however, that by rule 2

$$\ln(2x+7)^2 = 2 \cdot \ln(2x+7).$$

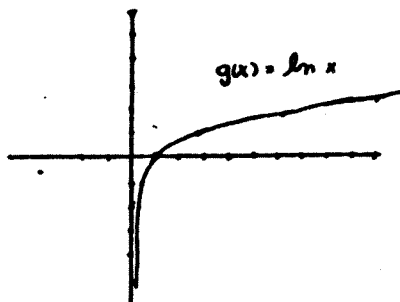
Now, by our definition, $\ln x$, for $x > 0$, is the area under the curve $1/x$ from 1 to x .



Note: if $0 < x < 1$ then $\ln x < 0$.

Why can't we use the same definition for $x < 0$? Because $1/x$ is undefined at 0, and we can't "cancel" \rightarrow and \leftarrow .

Using techniques of graphing we get:



Note that the area from 1 to x is different at every single x . So $f(x) = \ln x$ is one to one, and thus it has an inverse. We will use this fact in the section on the natural exponential.

II. DERIVATIVES WITH $\ln x$

Now that we have a new clearly defined function, the next question is, how do we differentiate it?

$$\frac{d}{dx} \ln x = \frac{d}{dx} \int_1^x \frac{1}{t} dt = \frac{1}{x} \quad \text{for } x > 0$$

fundamental theorem of calculus

With this you can do higher order derivatives. Thus

$$\frac{d^2}{dx^2} (\ln x) = \frac{d}{dx} \left(\frac{1}{x} \right) = \frac{-1}{x^2} \quad \text{etc.}$$

The product rule also works as usual, as does the quotient rule.

Example 1 $\frac{d}{dx} (5x \ln x) = 5x \cdot \frac{1}{x} + 5 \cdot \ln x = 5 + 5 \ln x$

Example 2 $\frac{d}{dx} \left(\frac{\ln x}{x} \right) = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}$

The chain rule also works as usual. If $u = f(x)$ what is $\frac{d}{dx} (\ln u)$?

Let $y = \ln u$.

$$\text{Then } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\text{so } \frac{d}{dx} (\ln u) = \frac{d}{dx} (y) = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \frac{du}{dx}$$

Example 3 $\frac{d}{dx} \ln (3x+5)$

let $u = 3x+5$ $y = \ln u$

$$\frac{du}{dx} = 3 \quad \frac{dy}{du} = \frac{1}{u}$$

so

$$\frac{d}{dx} \ln(3x+5) = \frac{d}{dx} (\ln u) = \frac{1}{u} \frac{du}{dx} = \frac{1}{3x+5} \cdot 3$$

Example 4 $\frac{d}{dx} \ln(5x+7x^2)$

let $u = 5x+7x^2$ $y = \ln u$

$\frac{du}{dx} = 5+14x$ $\frac{dy}{du} = \frac{1}{u}$

$\frac{d}{dx}(\ln(5x+7x^2)) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \frac{du}{dx} = \frac{1}{5x+7x^2} \cdot (5+14x)$

Example 5 $\frac{d}{dx} \ln(3x+5)^2$

Two methods.

A. let $u = (3x+5)^2$ $y = \ln u$

$\frac{du}{dx} = 2 \cdot (3x+5) \cdot 3$ $\frac{dy}{du} = \frac{1}{u}$

$\frac{d}{dx}(\ln u) = \frac{1}{u} \frac{du}{dx} = \frac{1}{(3x+5)^2} \cdot 2(3x+5) \cdot 3 = \frac{6}{3x+5}$

B. Simplify first $\ln(3x+5)^2 = 2 \cdot \ln(3x+5)$

$\frac{d}{dx} \ln(3x+5)^2 = 2 \cdot \frac{d}{dx} \ln(3x+5)$

let $u = 3x+5$ $y = \ln u$

$\frac{du}{dx} = 3$ $\frac{dy}{du} = \frac{1}{u}$

$2 \frac{d}{dx}(\ln u) = 2 \cdot \frac{1}{u} \frac{du}{dx} = 2 \cdot \frac{1}{3x+5} \cdot 3 = \frac{6}{3x+5}$

In complicated cases, method B (i.e., simplifying first) is easier.

III. INTEGRALS WITH LN X

Now if $\ln x = \int_1^x \frac{1}{t} dt$ what is a good way to write $\int \frac{1}{t} dt$?

$$\int \frac{1}{t} dt = \ln |t| + C$$

The $| |$ are important since \ln is not defined for negative numbers. Now, since

$$\int \frac{1}{t} dx = \ln|x| + C,$$

can we define $\frac{d}{dx} \ln|x|$?

if $x > 0$, then $\frac{d}{dx} \ln|x| = \frac{d}{dx} \ln x = \frac{1}{x}$

if $x < 0$, then $\frac{d}{dx} \ln|x| = \frac{d}{dx} \ln(-x) = \frac{1}{-x} \cdot -1 = \frac{1}{x}$

so $\frac{d}{dx} \ln|x| = \frac{1}{x}$.

What about $\int \frac{1}{f(x)} dx$?

Example 1 $\int \frac{dx}{3x+2}$ Here we integrate by substitution.

Let $u = 3x+5$

$du = 3dx$

$\frac{1}{3} du = dx$

$\int \frac{dx}{3x+5} = \int \frac{1}{u} \cdot \frac{1}{3} du = \frac{1}{3} \int \frac{du}{u} = \frac{1}{3} \ln|u| + C$

$= \frac{1}{3} \ln|3x+5| + C$

Example 2

$\int \frac{3x^2}{x^3+7} dx$

Note: the numerator is the derivative of the denominator.

let $u = x^3+7$

$du = 3x^2 dx$

$\int \frac{3x^2 dx}{x^3+7} = \int \frac{1}{u} \cdot du = \ln|u| + C = \ln|x^3+7| + C$

As you may have learned earlier, if the degree of the numerator is larger than the degree of the denominator, we can divide through by the denominator in hopes of getting something integrable.

Example 3. $\int \frac{3x^2+5x+1}{x+2} dx$

$$= \int \left(3x-1 + \frac{3}{x+2} \right) dx$$

$$= \int 3x dx + \int -1 dx + \int \frac{3}{x+2} dx$$

$$= \frac{3x^2}{2} - x + \int \frac{3}{x+2} dx$$

$$\begin{array}{r} 3x-1 \\ x+2 \overline{) 3x^2+5x+1} \\ \underline{3x^2+6x} \\ -x+1 \\ \underline{-x-2} \\ 3 \end{array}$$

For the last integral, let $u = x+2$
 $du = dx$

Then $\int \frac{3}{x+2} dx = \int \frac{3}{u} du = 3 \int \frac{1}{u} du = 3 \ln|u| + C = 3 \ln|x+2| + C,$

so

$$\int \frac{3x^2+5x+1}{x+2} dx = \frac{3x^2}{2} - x + 3 \ln|x+2| + C$$

There is one last kind of integral we now know how to do. We can't take $\int \ln x dx$ unless we know integration by parts, but we can take some integrals with $\ln x$ involved. For these we will again use u substitution.

Example 4 $\int \frac{1}{x \ln x} dx$

we must recognize one part as the derivative of the other.

let $u = \ln x$
 $du = \frac{1}{x} dx$

$$\int \frac{1}{x \ln x} dx = \int \frac{1}{u} du = \ln|u| + c = \ln|\ln x| + C$$

Example 5 $\int \frac{(\ln x)^2}{x} dx$

let $u = \ln x$
 $du = \frac{1}{x} dx$

$$\int \frac{(\ln x)^2}{x} dx = \int u^2 \cdot du = \frac{u^3}{3} + C = \frac{(\ln x)^3}{3} + C$$

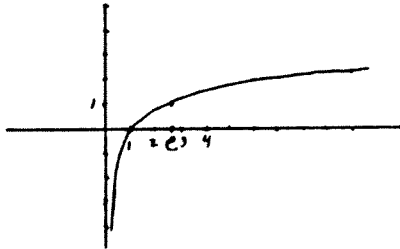
Problems with $\ln x$ (Solutions at back of capsule).

1. $\frac{d}{dx} \ln 3x$
2. $\frac{d}{dx} \ln(6x^2+5)$
3. $\frac{d}{dx} \frac{\ln(3x+5)}{2x+7}$
4. $\frac{d}{dx} \ln\left(\frac{4x+7}{2x+3}\right)$
5. $\frac{d}{dx} [(5x+9) \cdot \ln(3x^2+6)]$
6. $\frac{d}{dx} \ln(6x^3 + 12x^2 + 7)$
7. $\int \frac{1}{3x+7} dx$
8. $\int \frac{x dx}{x^2+4}$
9. $\int \frac{6x^2 + 7x + 3}{2x+5} dx$
10. $\int \frac{2x+4}{x^2 + 4x + 3} dx$
11. $\int \frac{\ln x}{x} dx$
12. $\int \frac{\ln(x+3)}{x+3} dx$
13. $\int \frac{1}{x(\ln x)^2} dx$

IV. EXPONENTIAL FUNCTIONS: e^x

This part of the capsule assumes you have seen exponential functions before, although it will give some basics. If you haven't seen exponential functions before you should get the precalculus capsule on EXPONENTS.

As we commented before $f(x) = \ln x$ is one-to-one; so $f(x) = \ln x$ has an inverse. Furthermore $f(x) = \ln x$ is 1-1 implies there is one and only one point x such that $\ln x = 1$. Call this point e .



Using the rules for the natural logarithm we get

$$\ln e^n = n \ln e = n \cdot 1 = n$$

Now since $\ln x$ is 1-1, we know that for all a the equation $\ln x = a$ has a unique solution, but $\ln e^a = a$, so that unique solution is $x = e^a$. So $\ln x = a \rightarrow x = e^a$. On the other hand

if $x = e^a$	then applying \ln to both sides
$\ln x = \ln e^a$	which implies
$\ln x = a \ln e$	which implies
$\ln x = a,$	

so $\boxed{\ln x = a \leftrightarrow x = e^a}$

Example:

$\ln x = 5$ defines a unique value of x ; just as $x+3 = 5$ does

$$\ln x = 5 \leftrightarrow x = e^5$$

We use the equivalence

$$\boxed{\ln x = y \leftrightarrow e^y = x}$$

to prove a number of statements in terms of variables, such as $g(x) = e^x$ is the inverse function of $f(x) = \ln x$

let $y = \ln x$ and thus $e^y = x$

Then $e^{\ln x} = e^y = x$

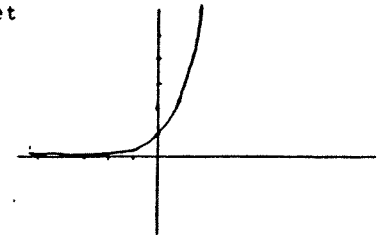
but we also know $\ln(e^x) = x \ln e = x$

so $g \circ f(x) = e^{\ln x} = x$

and $f \circ g(x) = \ln e^x = x$

Therefore $f(x) = \ln x$ and $g(x) = e^x$ are inverse functions

Now, using the fact that to graph the inverse of a function we flip it across the line $y = x$, we get



is the graph of the natural exponential.

What about derivatives and integrals? What is $\frac{d}{dx}(e^x)$?

This is a difficult question and requires a trick to solve. Since we already know how to differentiate logs, we write this as an equivalent problem in log form.

let $y = e^x$ so $\ln y = x$.

We are looking for $\frac{dy}{dx}$.

Differentiate both sides of $\ln y = x$ with respect to x

$$\frac{d}{dx} (\ln y) = \frac{d}{dx} x$$

$$\frac{1}{y} \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = y$$

but $y = e^x$

so $\boxed{\frac{d}{dx} e^x = e^x}$

a similar proof using the chain rule shows

$$\frac{d}{dx} e^u = e^u \frac{du}{dx} \quad \text{if } u = g(x)$$

What about $\int e^x dx$

Well since $\frac{d}{dx} e^x = e^x$

$$\int e^x dx = e^x + C$$

and just as we can expand the rule for derivatives by using the chain rule we can expand this one, using u -substitution.

Example 1

$$\int e^{3x} dx$$

$$\text{let } u = 3x$$

$$du = 3dx$$

$$\frac{du}{3} = dx$$

$$\int e^{3x} dx = \int e^u \cdot \frac{du}{3} = \frac{1}{3} \int e^u du = \frac{1}{3} e^u + C = \frac{1}{3} e^{3x} + C$$

Example 2

$$\int x e^{3x^2+5} dx$$

$$\text{let } u = 3x^2+5$$

$$du = 6x dx$$

$$\frac{du}{6} = x dx$$

$$\begin{aligned} \int e^{3x^2+5} \cdot x dx &= \int e^u \cdot \frac{du}{6} = \frac{1}{6} \int e^u du = \frac{1}{6} e^u + C \\ &= \frac{1}{6} e^{3x^2+5} + C \end{aligned}$$

in both these examples, and indeed in many problems, the u is substituted for the exponent. But alas, this is not always the case.

Example 3

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$$

here letting $u = x$ or $-x$ doesn't help one bit. if

$$u = \text{numerator}$$

$$u = e^x - e^{-x}$$

$$du = e^x + e^{-x} dx$$

We can't get a denominator. False start.

so try

$$u = \text{denominator}$$

$$u = e^x + e^{-x}$$

$$du = e^x - e^{-x} dx$$

yeh!

$$\begin{aligned} \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx &= \int \frac{du}{u} = \ln |u| + C \\ &= \ln |e^x + e^{-x}| + C \end{aligned}$$

(6)

Problems with e^x

(Solutions at back of capsule.)

1. $\frac{d}{dx} (e^{3x^2+2}) =$
2. $\frac{d}{dx} \left(\frac{x^2}{e^x}\right) =$
3. $\frac{d}{dx} (e^{\sqrt{x}} + \sqrt{e^x}) =$
4. $\frac{d}{dx} [(e^{2x} - e^{4x})^3]$
5. $\frac{d}{dx} \left(\frac{e^x}{x^2+1}\right)$
6. $\frac{d}{dx} \frac{\ln(e^x+1)}{e^x+1}$
7. $\int \frac{e^{2/x}}{x^2} dx$
8. $\int \frac{e^x}{e^x+1} dx$
9. $\int \frac{2e^{2x} + 3e^{3x} - e^{-x}}{e^{2x} + e^{3x} + e^{-x}} dx$
10. $\int (e^{-4x}+1) dx$
11. $\int \frac{e^{x^{1/3}}}{x^{2/3}} dx$

V. a^x AND $\log_a x$

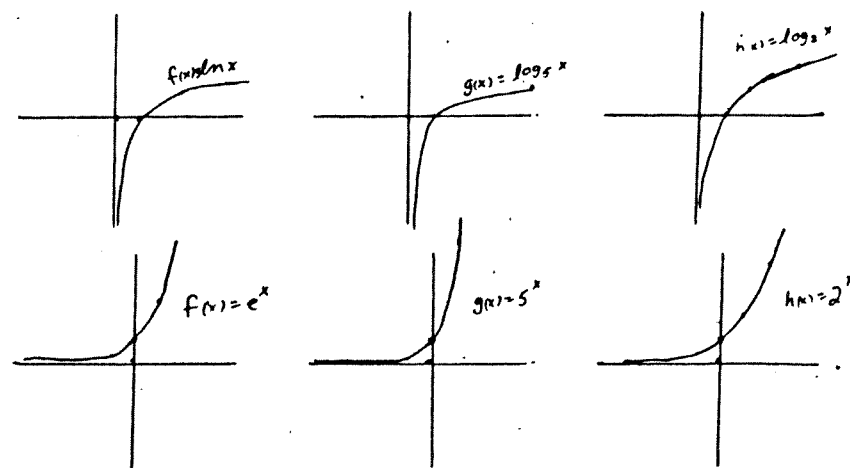
Now that we have handled e^x and $\ln x$, what about a^x and $\log_a x$?

We have

$e^x = y$	$\ln y = x$
$a^x = y$	$\log_a y = x$

Just as $f(x) = e^x$ and $g(x) = \ln x$ are inverses, it can be shown $h(x) = a^x$ and $j(x) = \log_a x$ are inverses.

The graphs of a^x look much like e^x and the graphs of $\log_a x$ look much like $\ln x$



Now that we have an idea of what the functions a^x and $\log_a x$ are, what are their derivatives and integrals?

What is $\frac{d}{dx} a^x$?

We will do this in a roundabout method, by noting

$$a^x = e^{\ln a^x} = e^{x \ln a}$$

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = \ln a \cdot e^{x \ln a} = \ln a \cdot a^x = a^x \ln a$$

likewise

$$\boxed{\frac{d}{dx} a^u = a^u \cdot \ln a \cdot \frac{du}{dx}}$$

It can also be shown

$$\int a^x dx = \int e^{x \ln a} dx = \frac{1}{\ln a} \cdot e^{x \ln a} + C = \frac{a^x}{\ln a} + C$$

Note-- the proof for both of these statements involved converting to natural exponential. In practice we can do one of two methods - 1. use the rules for a^x . 2. convert to $e^{x \ln a}$ and proceed as before.

Example 1

$$\int 3^x dx = \frac{3^x}{\ln 3} + C$$

Example 2

$$\int 3^x dx = \int e^{x \ln 3} dx$$

$$\text{Let } u = x \ln 3$$

$$du = \ln 3 \cdot dx$$

$$\frac{du}{\ln 3} = dx$$

$$\int e^u \frac{du}{\ln 3} = \frac{1}{\ln 3} \cdot \int e^u du = \frac{1}{\ln 3} e^u + C$$

$$= \frac{1}{\ln 3} e^{x \ln 3} + C = \frac{3^x}{\ln 3} + C$$

Now what about $\frac{d}{dx} \log_a x$?

For this we will use another roundabout method. We will change to natural logarithms.

$$\log_a x = y \Leftrightarrow a^y = x$$

$$\Leftrightarrow \ln a^y = \ln x$$

$$\Leftrightarrow y \ln a = \ln x$$

$$\Leftrightarrow y = \frac{\ln x}{\ln a}$$

$$\text{so } \log_a x = \frac{\ln x}{\ln a}$$

$$\text{so } \frac{d}{dx} (\log_a x) = \frac{d}{dx} \left(\frac{\ln x}{\ln a} \right) = \frac{1}{\ln a} \cdot \frac{d}{dx} \ln x = \frac{1}{x \ln a}$$

now $\int \log_a x dx$, like $\int \ln x dx$, is undoable as yet (unless you know integration by parts) but we can integrate functions like

$$\int \frac{1}{x \log_a x} dx \text{ just as we can } \int \frac{1}{x \ln x} dx$$

Example 3

$$\int \frac{1}{x \log_a x} dx \text{ ---- two methods}$$

A. use $\log_a x = \frac{\ln x}{\ln a}$

$$\int \frac{1}{x \log_a x} dx = \int \frac{\ln a dx}{x \ln x} = \ln a \int \frac{1}{x \ln x} dx$$

do u substitution as before

$$\text{Let } u = \ln x$$

$$\text{Then } du = (1/x) dx$$

$$\text{So our integral} = \ln a \int \frac{1}{u} du = (\ln a) \cdot (\ln u) + C = (\ln a)(\ln(\ln x)) + C$$

$$8. \int \frac{1}{x \log_a x} dx$$

$$\text{let } u = \log_a x$$

$$du = \frac{1}{x \ln a} dx$$

$$\ln a \, du = \frac{1}{x} dx$$

$$= \int \frac{\ln a \, du}{u} = \ln a \int \frac{du}{u} = (\ln a)(\ln(\ln x)) + C$$

↑ finish as above

Further examples:

$$4. \text{ if } y = 3^{\sqrt{x}}, \text{ what is } \frac{dy}{dx}?$$

$$\frac{dy}{dx} = 3^{\sqrt{x}} \cdot \ln 3 \cdot \frac{d(\sqrt{x})}{dx} = 3^{\sqrt{x}} \cdot \ln 3 \cdot \frac{1}{2} \cdot x^{-1/2}$$

$$5. \frac{d}{dx} (5^{x^2+2x}) = 5^{x^2+2x} \cdot \ln 5 \cdot \frac{d}{dx} (x^2+2x)$$

$$= 5^{x^2+2x} \cdot \ln 5 \cdot (2x+2)$$

$$6. \int 2^{3x-1} dx$$

$$\text{let } u = 3x-1$$

$$du = 3 dx$$

$$\frac{du}{3} = dx$$

$$= \int 2^u \cdot \frac{du}{3} = \frac{1}{3} \int 2^u du = \frac{1}{3} \cdot \frac{2^u}{\ln 2} + C = \frac{1}{3} \cdot \frac{2^{3x-1}}{\ln 2} + C$$

Remember, if you blank out on the new rules for a^x and $\log_a x$ you can switch to $e^x \ln a$ and $\frac{\ln x}{\ln a}$ and use the old rules.

Problems with a^x and $\log_a x$

(Solutions at back of capsule.)

$$1. \frac{d}{dx} (3^{x^2+1}) =$$

$$2. \frac{d}{dx} (5^{\sqrt{x}}) =$$

$$3. - \frac{d}{dx} (\log_5 \sqrt{x^2+1}) =$$

$$4. \frac{d}{dx} (x^3+3^x) =$$

$$5. \int \frac{3^x}{\sqrt{3^x+4}} dx =$$

$$6. \int 3^{-2x} dx =$$

$$7. \int \frac{1}{x(\log_2 x)^2} dx =$$

$$8. \int \frac{\log_3 x}{x} dx =$$

Solutions to Problems on $\ln x$

1. $\frac{d}{dx} \ln 3x = \frac{1}{3x} \cdot \frac{d}{dx}(3x) = \frac{1}{3x} \cdot 3 = \frac{1}{x}$

2. $\frac{d}{dx} \ln(6x^2+5) = \frac{1}{6x^2+5} \cdot \frac{d}{dx}(6x^2+5) = \frac{1}{6x^2+5} \cdot 12x$

3. $\frac{d}{dx} \frac{\ln(3x+5)}{2x+7} = \frac{(2x+7) \cdot \frac{1}{3x+5} \frac{d}{dx}(3x+5) - \ln(3x+5) \cdot 2}{(2x+7)^2}$
 $= \frac{(2x+7)(\frac{3}{3x+5}) - 2 \ln(3x+5)}{(2x+7)^2}$

4. (a) $\frac{d}{dx} (\ln(\frac{4x+7}{2x+3})) = \frac{d}{dx} (\ln(4x+7)) - \ln(2x+3)$
 $= \frac{1}{4x+7} \cdot \frac{d}{dx}(4x+7) - \frac{1}{2x+3} \cdot \frac{d}{dx}(2x+3)$
 $= \frac{4}{4x+7} - \frac{2}{2x+3}$

If you didn't simplify first you would get a more complicated but identical answer.

(b) $\frac{d}{dx} \ln(\frac{4x+7}{2x+3}) = \frac{2x+3}{4x+7} \cdot \frac{d}{dx} (\frac{4x+7}{2x+3})$
 $= \frac{2x+3}{4x+7} \cdot \frac{(2x+3) \cdot 4 - (4x+7) \cdot 2}{(2x+3)^2}$

5. $\frac{d}{dx} [(5x+9)(\ln(3x^2+6))] = 5x+9 \cdot \frac{1}{3x^2+6} \cdot \frac{d}{dx}(3x^2+6) + 5 \ln(3x^2+6)$
 $= \frac{5x+9}{3x^2+6} \cdot 6x + 5 \ln(3x^2+6)$

6. $\frac{d}{dx} \ln(6x^3+12x^2+7) = \frac{1}{6x^3+12x^2+7} \cdot \frac{d}{dx}(6x^3+12x^2+7)$
 $= \frac{1}{6x^3+12x^2+7} \cdot (18x^2+24x)$

7. $\int \frac{1}{3x+7} dx$ let $u = 3x+7$
 $du = 3dx$
 $\frac{du}{3} = dx$

$\int \frac{1}{u} \cdot \frac{du}{3} = \frac{1}{3} \int \frac{du}{u} = \frac{1}{3} \ln |u| + C$
 $= \frac{1}{3} \ln |3x+7| + C$

8. $\int \frac{x dx}{x^2+4}$ let $u = x^2+4$
 $du = 2x dx$
 $du/2 = x dx$

$\int \frac{1}{u} \cdot \frac{du}{2} = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| + C$
 $= \frac{1}{2} \ln |x^2+4| + C$

9. $\int \frac{5x^2+7x+3}{2x+5} dx$ $2x+5 \sqrt{\frac{3x-4}{6x^2+7x+3}}$
 $\frac{6x^2+15x}{-8x+3}$
 $\frac{-8x-20}{23}$

$\int (3x-4 + \frac{23}{2x+5}) dx$
 $= \frac{3x^2}{2} - 4x + 23 \int \frac{dx}{2x+5}$ let $u = 2x+5$
 $du = 2dx$
 $\frac{du}{2} = dx$
 $= \frac{3x^2}{2} - 4x + 23 \int \frac{1}{u} \cdot \frac{du}{2} = \frac{3x^2}{2} - 4x + \frac{23}{2} \ln |u| + C$
 $= \frac{3x^2}{2} - 4x + \frac{23}{2} \ln |2x+5| + C$

10. $\int \frac{2x+4}{x^2+4x+3} dx$ let $u = x^2+4x+3$
 $du = (2x+4) dx$

$\int \frac{1}{u} du = \ln |u| + C = \ln |x^2+4x+3| + C$

$$11. \int \frac{\ln x}{x} dx \quad \text{let } u = \ln x \\ du = \frac{1}{x} dx$$

$$\int u du = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C$$

$$12. \int \frac{\ln(x+3)}{x+3} dx \quad u = \ln(x+3) \\ du = \frac{1}{x+3} dx$$

$$\int u du = \frac{u^2}{2} + C = \frac{[\ln(x+3)]^2}{2} + C$$

$$13. \int \frac{1}{x(\ln x)} dx \quad u = \ln x \\ du = \frac{1}{x} dx$$

$$\int \frac{1}{u^2} du = \int u^{-2} du = -u^{-1} + C = -(\ln x)^{-1} + C = \frac{-1}{\ln x} + C$$

Solutions to problems on e^x

$$1. \frac{d}{dx} (e^{3x^2+2}) = e^{3x^2+2} \cdot \frac{d}{dx} (3x^2+2) = e^{3x^2+2} \cdot 6x$$

$$2. \frac{d}{dx} \left(\frac{x^2}{e^x} \right) = \frac{e^x \cdot 2x - x^2 \cdot e^x}{(e^x)^2}$$

$$3. \frac{d}{dx} (e^{\sqrt{x}} + \sqrt{e^x}) = e^{\sqrt{x}} \cdot \frac{d}{dx} \sqrt{x} + \frac{1}{2} (e^x)^{-1/2} \cdot \frac{d}{dx} e^x \\ = e^{\sqrt{x}} \cdot \frac{1}{2} x^{-1/2} + \frac{1}{2} (e^x)^{-1/2} \cdot e^x$$

$$4. \frac{d}{dx} [(e^{2x} - e^{4x})^3] = 3(e^{2x} - e^{4x})^2 \cdot \frac{d}{dx} (e^{2x} - e^{4x}) \\ = 3(e^{2x} - e^{4x})^2 \cdot (2e^{2x} - 4e^{4x})$$

$$5. \frac{d}{dx} \left(\frac{e^x}{x^2+1} \right) = \frac{(x^2+1)e^x - e^x(2x)}{(x^2+1)^2}$$

$$6. \frac{d}{dx} \frac{\ln(e^x+1)}{e^x+1} = \frac{(e^x+1) \cdot \frac{1}{e^x+1} \cdot e^x - e^x \ln(e^x+1)}{(e^x+1)^2} = \frac{e^x - e^x \ln(e^x+1)}{(e^x+1)^2}$$

$$7. \int \frac{e^{2/x}}{x^2} dx \quad u = 2/x \\ du = -\frac{2}{x^2} dx \\ \frac{-du}{2} = \frac{1}{x^2} dx$$

$$\int e^u \cdot -\frac{du}{2} = -\frac{1}{2} \int e^u du = -\frac{1}{2} e^u + C = -\frac{1}{2} e^{2/x} + C$$

$$8. \int \frac{e^x}{e^x+1} dx \quad u = e^x+1 \\ du = e^x dx$$

$$\int \frac{1}{u} du = \ln |u| + C = \ln |e^x+1| + C$$

$$9. \int \frac{2e^{2x} + 3e^{3x} - e^{-x}}{e^{2x} + e^{3x} + e^{-x}} dx \quad u = e^{2x} + e^{3x} + e^{-x} \\ du = 2e^{2x} + 3e^{3x} - e^{-x} dx$$

$$= \int \frac{1}{u} du = \ln |u| + C = \ln |e^{2x} + e^{3x} + e^{-x}| + C$$

$$10. \int (e^{-4x} + 1) dx = \int e^{-4x} dx + \int dx = \int e^{-4x} dx + x + C$$

$$u = -4x \quad \int e^u \cdot \frac{du}{-4} + x + C$$

$$du = -4dx$$

$$= \frac{-1}{4} \int e^u du + x + C$$

$$\frac{du}{-4} = dx$$

$$= \frac{-1}{4} e^u + x + C = \frac{-1}{4} e^{-4x} + x + C$$

$$11. \int \frac{e^{x^{1/3}}}{x^{2/3}} dx$$

$$u = x^{1/3}$$

$$du = \frac{1}{3} x^{-2/3} dx$$

$$3du = \frac{1}{x^{2/3}} dx$$

$$\int e^u \cdot 3du = 3 \int e^u du = 3e^u + C = 3e^{x^{1/3}} + C$$

Solutions to problems involving a^x and $\log_a x$

1. $\frac{d}{dx} (3^{x^2+1}) = 3^{x^2+1} \cdot \ln 3 \cdot \frac{d}{dx} (x^2+1) = (2x)(\ln 3) \cdot 3^{x^2+1}$

2. $\frac{d}{dx} 5^{\sqrt{x}} = 5^{\sqrt{x}} \ln 5 \cdot \frac{d}{dx} \sqrt{x} = 5^{\sqrt{x}} \ln 5 \cdot \frac{1}{2} x^{-1/2}$

3. $\frac{d}{dx} (\log_5 \sqrt{x^2+1}) = \frac{1}{\sqrt{x^2+1} \cdot \ln 5} \cdot \frac{d}{dx} \sqrt{x^2+1}$
 $= \frac{1}{\ln 5 \cdot \sqrt{x^2+1}} \cdot \frac{1}{2} (x^2+1)^{-1/2} \cdot 2x$

4. $\frac{d}{dx} (x^3+3^x) = 3x^2 + 3^x \ln 3$

5. $\int \frac{3^x}{\sqrt{3^x+4}} dx$ let $u = 3^x+4$
 $du = 3^x \ln 3 dx$
 $\frac{du}{\ln 3} = 3^x dx$

$\int \frac{1}{\sqrt{u}} \cdot \frac{du}{\ln 3} = \frac{1}{\ln 3} \int u^{-1/2} du = \frac{1}{\ln 3} \cdot 2u^{1/2} + C$
 $= \frac{2}{\ln 3} \cdot (3^x+4)^{1/2} + C$

6. $\int 3^{-2x} dx$ $u = -2x$
 $du = -2dx$
 $\frac{du}{-2} = dx$

$\int 3^u \cdot \frac{du}{-2} = -\frac{1}{2} \int 3^u du = -\frac{1}{2} \frac{3^u}{\ln 3} + C = -\frac{1}{2} \frac{3^{-2x}}{\ln 3} + C$

7. $\int \frac{1}{x(\log_2 x)^2} dx$

$u = \log_2 x$ $du = \frac{1}{x \ln 2} dx$ $\ln 2 \cdot du = \frac{1}{x} dx$
--

$\int \frac{1}{u^2} \cdot \ln 2 du = \ln 2 \cdot \int u^{-2} du$
 $= \ln 2 \cdot (-u^{-1}) + C$
 $= -\frac{\ln 2}{u} + C = -\frac{\ln 2}{\log_2 x} + C$

[can also do by using $\log_2 x = \frac{\ln x}{\ln 2}$]

8. $\int \frac{\log_3 x}{x} dx$

let $u = \log_3 x$
 $du = \frac{1}{x \ln 3} dx$
 $\ln 3 du = \frac{1}{x} dx$

$\int u \cdot \ln 3 du = \ln 3 \int u du = (\ln 3) \cdot \frac{u^2}{2} + C$
 $= \frac{(\ln 3)(\log_3 x)^2}{2} + C$

[again can also do by letting $\log_3 x = \frac{\ln x}{\ln 3}$]