Inverse Trig Functions

March 2, 2009

Contents

1 Introduction

Just as trig functions arise in many applications, so do the inverse trig functions. What may be most surprising is that the inverse trig functions give us solutions to some common integrals. For example, suppose you need to evaluate the integral

$$\int_a^b \frac{1}{\sqrt{1-x^2}}\ dx$$

for some appropriate values of a and b. You can use the inverse sine function to solve it! In this capsule we do not attempt to derive the formulas that we use; you should look at your textbook for derivations and complete explanations. This material simply summarizes the key results and gives examples of how to use them. As usual, all angles used here are in radians.

2 Restrictions on the Domains of the Trig Functions

A function must be one-to-one for it to have an inverse. The trig functions are not one-to-one and in fact are periodic (i.e. their values repeat themselves periodically). In order to define inverse functions we need to restrict the domain of each trig function to a region in which it is one-to-one and also attains all of its values. We do this by selecting a specific period for each function and using this period as a restricted domain on which an inverse function can be defined. There are an infinite number of different restrictions we could chose, but the following are the ones that are normally used.

Standard Restricted Domains for Trig Functions		
Function	**Domain**	**Range**
$sin(x)$	$[\frac{-\pi}{2}, \frac{\pi}{2}]$	$[-1, 1]$
$cos(x)$	$[0, \pi]$	$[-1, 1]$
$tan(x)$	$(\frac{-\pi}{2}, \frac{\pi}{2})$	$(-\infty, \infty)$
$cot(x)$	$(0, \pi)$	$(-\infty, \infty)$
$sec(x)$	$[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$	$(-\infty, -1] \cup [1, \infty)$
$csc(x)$	$[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$	$(-\infty, -1] \cup [1, \infty)$

3 Definitions of the Inverse Functions

When the trig functions are restricted to the domains above they become one-to-one functions and we can define the inverse functions. For the sine function we use the notation $sin^{-1}(x)$ or $arcsin(x)$ to denote the inverse function. Both are read "arc sine". Look carefully at where we have placed the -1. Written this way it indicates the *inverse of the sine function*. If, instead, we write $(sin(x))^{-1}$ we mean the fraction $\frac{1}{sin(x)}$. The other inverse functions are denoted in a similar way.

The following table summarizes the domains and ranges of the inverse trig functions. Note that for each inverse trig function we have swapped the domain and range of the corresponding trig function.

Standard Domains and Ranges for Inverse Trig Functions		
Function	**Domain**	**Range**
$sin^{-1}(x)$	$[-1, 1]$	$[\frac{-\pi}{2}, \frac{\pi}{2}]$
$cos^{-1}(x)$	$[-1, 1]$	$[0, \pi]$
$tan^{-1}(x)$	$(-\infty, \infty)$	$(\frac{-\pi}{2}, \frac{\pi}{2})$
$cot^{-1}(x)$	$(-\infty, \infty)$	$(0, \pi)$
$sec^{-1}(x)$	$(-\infty, -1] \cup [1, \infty)$	$[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$
$csc^{-1}(x)$	$(-\infty, -1] \cup [1, \infty)$	$[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$

Now define the *arcsin* function by

$$y = sin^{-1}(x) \text{ if and only if } x \text{ is in } [-1,\ 1],\ y \text{ is in } [\tfrac{-\pi}{2}, \tfrac{\pi}{2}], \text{ and } sin(y) = x$$

Note that $sin^{-1}(x)$ is only defined when x is in the interval $[-1, 1]$. The other inverse functions are defined similarly, using the corresponding trig functions and their restricted domains.

4 Some Useful Identities

Here are a few identities you may find helpful.

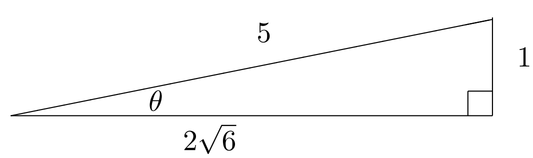
$cos^{-1}(x) + cos^{-1}(-x) = \pi$

$sin^{-1}(x) + cos^{-1}(x) = \frac{\pi}{2}$

$tan^{-1}(-x) = -tan^{-1}(x)$

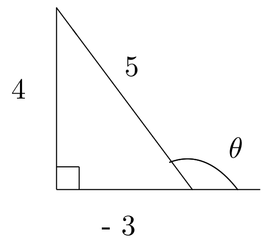
5 Practicing with the Inverse Functions

Example 1: Find the value of $tan(sin^{-1}(\frac{1}{5}))$.
Solution: The best way to solve this problem is to draw a triangle and use the Pythagorian Theorem.



Here we let θ represent the value of $sin^{-1}(\frac{1}{5})$. Label the hypotenuse and the side opposite θ by using the value of the sin of the angle θ. Next use the Pythagorian Theorem to get the remaining side. You now have the information that is needed to find $tan(\theta)$. Since $tan(\theta) = \frac{opposite}{adjacent}$, the answer is $\frac{1}{\sqrt{24}} = \frac{1}{2\sqrt{6}}$

Example 2: Find the value of $sin(cos^{-1}(-\frac{3}{5}))$.
Solution: Look at the following picture:



In this picture we let $\theta = cos^{-1}(-\frac{3}{5})$. Then $0 \leq \theta \leq \pi$ and $cos\theta = -\frac{3}{5}$. Because $cos(\theta)$ is negative, θ must be in the second quadrant, i.e. $\frac{\pi}{2} \leq \theta \leq \pi$. Using the Pythagorean Theorem and the fact that θ is in the second quadrant we get that $sin(\theta) = \frac{\sqrt{5^2-3^2}}{5} = \frac{\sqrt{25-9}}{5} = \frac{4}{5}$. Note that although θ does not lie in the restricted domain used to make $sin(x)$ one-to-one, the unrestricted sin function is defined in the second quadrant and so we are free to use this fact.

6 Derivatives of Inverse Trig Functions

The derivatives of the inverse trig functions are shown in the following table.

Derivatives	
Function	**Derivative**
$sin^{-1}(x)$	$\frac{d}{dx}(sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}, \quad \lvert x\rvert < 1$
$cos^{-1}(x)$	$\frac{d}{dx}(cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}, \quad \lvert x\rvert < 1$
$tan^{-1}(x)$	$\frac{d}{dx}(tan^{-1}x) = \frac{1}{1+x^2}$
$cot^{-1}(x)$	$\frac{d}{dx}(cot^{-1}x) = \frac{-1}{1+x^2}$
$sec^{-1}(x)$	$\frac{d}{dx}(sec^{-1}x) = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}, \quad \lvert x\rvert > 1$
$csc^{-1}(x)$	$\frac{d}{dx}(csc^{-1}x) = \frac{-1}{\lvert x\rvert\sqrt{x^2-1}}, \quad \lvert x\rvert > 1$

In practice we are often interested in calculating the derivatives when the variable x is replaced by a function $u(x)$. This requires the use of the $c$hain rule. For example,

$$\frac{d}{dx}(sin^{-1}u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} = \frac{\frac{du}{dx}}{\sqrt{1-u^2}}, \quad |u| < 1$$

The other functions are handled in a similar way.

Example 1: Find the derivative of $y = cos^{-1}(x^3)$ for $|x^3| < 1$

Solution: Note that $|x^3| < 1$ if and only if $|x| < 1$, so the derivative is defined whenever $|x| < 1$.

$$\begin{aligned}\frac{d}{dx}(cos^{-1}(x^3)) &= -\frac{1}{\sqrt{1-(x^3)^2}} \cdot \frac{d}{dx}(x^3) \\ &= -\frac{1}{\sqrt{1-(x^3)^2}} \cdot (3x^2) \\ &= -\frac{3x^2}{\sqrt{1-x^6}}\end{aligned}$$

Example 2: Find the derivative of $y = tan^{-1}(\sqrt{3x})$.

Solution:

$$\begin{aligned}\frac{d}{dx}(tan^{-1}(\sqrt{3x})) &= \frac{1}{1+(\sqrt{3x})^2} \cdot \frac{d}{dx}(\sqrt{3x}) \\ &= \frac{1}{1+(\sqrt{3x})^2} \cdot \frac{1}{2\sqrt{3x}} \cdot 3 \\ &= \frac{3}{2\sqrt{3x}\ (1+3x)}\end{aligned}$$

Exercise 1: For each of the following, find the derivative of the given function with respect to the independent variable.

(a) $y = tan^{-1}\ t^4$

(b) $z = t\ cot^{-1}(1+t^2)$

(c) $x = sin^{-1}\sqrt{1-t^4}$

(d) $s = \frac{t}{\sqrt{1-t^2}}\ +\ cos^{-1}t$

(e) $y = sin^{-1}\sqrt{x}$

(f) $z = cot^{-1}(\frac{y}{1-y^2})$

Solutions:

(a) $y = tan^{-1}\ t^4$

$$\begin{aligned}\frac{dy}{dt} &= \frac{d}{dt}tan^{-1}\ (t^4)\\ &= \frac{1}{1+(t^4)^2} \cdot \frac{d}{dt}\ (t^4)\\ &= \frac{4t^3}{1+t^8}\end{aligned}$$

(b) $z = t\ cot^{-1}(1+t^2)$

$$\begin{aligned}\frac{dz}{dt} &= \frac{d}{dt}\ t\ cot^{-1}(1+t^2)\\ &= cot^{-1}(1+t^2) +\ t \cdot \frac{-1}{1+(1+t^2)^2} \cdot (2t)\\ &= cot^{-1}(1+t^2) - \frac{2t^2}{t^4+2t^2+2}\end{aligned}$$

(c) $x = sin^{-1}\sqrt{1-t^4}$

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt}\ sin^{-1}\sqrt{1-t^4}\\ &= \frac{1}{\sqrt{1-(\sqrt{1-t^4})^2}} \cdot \frac{d}{dt}(\sqrt{1-t^4})\\ &= \frac{1}{\sqrt{1-(1-t^4)}} \cdot \frac{1}{2} \cdot (1-t^4)^{\frac{-1}{2}} \cdot (-4t^3)\\ &= \frac{1}{\sqrt{1-1+t^4}} \cdot \frac{1}{\sqrt{1-t^4}} \cdot (-2t^3)\\ &= \frac{1}{t^2} \cdot \frac{1}{\sqrt{1-t^4}} \cdot (-2t^3)\\ &= \frac{-2t}{\sqrt{1-t^4}}\end{aligned}$$

(d) $s = \frac{t}{\sqrt{1-t^2}} \ + \ cos^{-1}t$

$$
\begin{aligned}
\frac{ds}{dt} &= \frac{d}{dt}\ \frac{t}{\sqrt{1-t^2}} \ + \ \frac{d}{dt}\ cos^{-1}t \\
&= \frac{(\sqrt{1-t^2}) \cdot 1 - t \cdot \frac{1}{2}(1-t^2)^{\frac{-1}{2}} \cdot (-2t)}{(\sqrt{1-t^2})^2} + \frac{-1}{\sqrt{1-t^2}} \\
&= \frac{\sqrt{1-t^2} + \frac{t^2}{\sqrt{1-t^2}}}{(1-t^2)} \ - \ \frac{1}{\sqrt{1-t^2}} \\
&= \frac{(\sqrt{1-t^2})(\sqrt{1-t^2}) + t^2}{(\sqrt{1-t^2})(1-t^2)} \ - \ \frac{(1-t^2)}{(1-t^2)} \cdot \frac{1}{(\sqrt{1-t^2})} \\
&= \frac{(1-t^2) + t^2 - (1-t^2)}{(\sqrt{1-t^2})(1-t^2)} \\
&= \frac{t^2}{(1-t^2)^{\frac{3}{2}}}
\end{aligned}
$$

(e) $y = sin^{-1}\sqrt{x}$

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\ sin^{-1}\sqrt{x} \\
&= \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{d}{dx}\ \sqrt{x} \\
&= \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2}x^{\frac{-1}{2}} \\
&= \frac{1}{2\sqrt{x(1-x)}}
\end{aligned}
$$

(f) $z = cot^{-1}(\frac{y}{1-y^2})$

$$\frac{dz}{dy} = \frac{d}{dy}cot^{-1}(\frac{y}{1-y^2})$$

$$= \frac{-1}{1+(\frac{y}{1-y^2})^2} \cdot \frac{d}{dy}(\frac{y}{1-y^2})$$

$$= \frac{-1}{\frac{(1-y^2)^2+y^2}{(1-y^2)^2}} \cdot \frac{d}{dy}(\frac{y}{1-y^2})$$

$$= \frac{-(1-y^2)^2}{(1-y^2)^2+y^2} \cdot \frac{(1-y^2) \cdot 1 - y \cdot (-2y)}{(1-y^2)^2}$$

$$= \frac{-1}{(1-y^2)^2+y^2} \cdot \frac{(1-y^2) \cdot 1 - y \cdot (-2y)}{1}$$

$$= \frac{-1(1-y^2+2y^2)}{1-2y^2+y^4+y^2}$$

$$= \frac{-(1+y^2)}{1-y^2+y^4}$$

7 Solving Integrals

The formulas given for the derivatives lead us to nice ways to solve some common integrals. The following is a list of useful ones. These formulas hold for constants $a \neq 0$

$$\int \frac{du}{\sqrt{a^2-u^2}} = sin^{-1}(\frac{u}{a}) + C \qquad \text{for } u^2 < a^2$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}tan^{-1}(\frac{u}{a}) + C \qquad \text{for all } u$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}sec^{-1}|\frac{u}{a}| + C \qquad \text{for } |u| > a > 0$$

Exercise 2: Verify each of the equations above by taking the derivative of the right hand side.

We now want to use these formulas to solve some common integrals.

Example 1: Evaluate the integral $\int \frac{dx}{\sqrt{9-16x^2}}$
Solution: Let $a = 3$ and $u = 4x$. Then $16x^2 = (4x)^2 = u^2$ and $du = 4dx$. We get the following for $16x^2 < 9$:

$$\int \frac{dx}{\sqrt{9-16x^2}} = \frac{1}{4}\int \frac{du}{\sqrt{a^2-u^2}}$$

$$= \frac{1}{4} sin^{-1}(\frac{u}{a}) + C$$

$$= \frac{1}{4} sin^{-1}(\frac{4x}{3}) + C$$

$$= \frac{1}{4} sin^{-1}(\frac{4}{3}x) + C$$

Exercise 3: Evaluate the following integrals.

(a) $\int \frac{dx}{\sqrt{25-4x^2}}$

(b) $\int \frac{dy}{36+4y^2}$

(c) $\int \frac{z\ dz}{5+2z^4}$

(d) $\int \frac{sin\,x\ dx}{\sqrt{10-cos^2 x}}$

(e) $\int \frac{dx}{\sqrt{5+4x-x^2}}$

(f) $\int \frac{7\ dx}{25-12x+4x^2}$

Solutions:

(a) $\int \frac{dx}{\sqrt{25-4x^2}}$. For this problem use the formula $\int \frac{du}{\sqrt{a^2-u^2}} = sin^{-1}\frac{u}{a} + C$ with $a = 5$, $u = 2x$ and $du = 2\,dx$, giving you $\int \frac{dx}{\sqrt{25-4x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{a^2-u^2}} = \frac{1}{2} sin^{-1}(\frac{2x}{5}) + C$

(b) $\int \frac{dy}{36+4y^2}$. Use the formula $\int \frac{du}{a^2+u^2} = \frac{1}{a} tan^{-1}(\frac{u}{a}) + C$ with $a = 6$, $u = 2y$ and $du = 2\,dy$.

This gives us $\int \frac{dy}{36+4y^2} = \frac{1}{2}\int \frac{du}{a^2+u^2} = (\frac{1}{2})(\frac{1}{6})\,tan^{-1}(\frac{2y}{6}) + C = \frac{1}{12}\,tan^{-1}(\frac{y}{3}) + C$

(c) $\int \frac{z\ dz}{5+2z^4}$. In order to make the calculations a bit simpler, it is useful to multiply the numerator and denominator by 2 in order to get the term $4z^4$ instead of $2z^4$ in the denominator. This gives us $\int \frac{z\ dz}{5+2z^4} = \int \frac{2z\ dz}{10+4z^4}$.
Now let $u = 2z^2$, $du = 4z\,dz$ and $a = \sqrt{10}$ and we have

$$\int \frac{z\,dz}{5+2z^4} = \frac{1}{2}\int \frac{4\,z\,dz}{10+4z^4} = \frac{1}{2}\int \frac{du}{(\sqrt{10})^2+u^2} = \frac{1}{2\sqrt{10}}\ tan^{-1}(\frac{2z^2}{\sqrt{10}}) + C$$

(d) $\int \frac{sin\,x\ dx}{\sqrt{10-cos^2 x}}$. Let $u = cos\,x$, $du = -sin\,x\,dx$ and $a = \sqrt{10}$. Then

$$\int \frac{sin\,x\ dx}{\sqrt{10-cos^2 x}} = (-1)\int \frac{-\,sin\,x\,dx}{\sqrt{10-cos^2 x}} = -\,sin^{-1}(\frac{cos\,x}{\sqrt{10}}) + C$$

(e) $\int \frac{dx}{\sqrt{5+4x-x^2}}$. Transform this expression into something with the form $\int \frac{du}{\sqrt{a^2-u^2}}$. To do this we need to complete the square of the expression in the denominator as follows:

$$\begin{aligned} 5+4x-x^2 &= 5+4-4+4x-x^2 \\ &= 9-4+4x-x^2 \\ &= 9-(x^2-4x+4) \\ &= (3)^2-(x-2)^2 \end{aligned}$$

This gives us

$$\int \frac{dx}{\sqrt{5+4x-x^2}} \;=\; \int \frac{dx}{\sqrt{(3)^2-(x-2)^2}} \;=\; sin^{-1}(\tfrac{x-2}{3}) \;+\; C$$

(f) $\int \frac{7\ dx}{25-12x+4x^2}$. Complete the square and transform the expression into something with the form $\int \frac{du}{a^2+u^2}$. Rewrite the denominator as follows:

$$\begin{aligned} 25-12x+4x^2 &= 16+9-12x+4x^2 \\ &= (4)^2+(2x-3)^2 \end{aligned}$$

Now, letting $u \;=\; 2x-3$ and $du \;= 2\,dx$ we get

$$\int \frac{7\ dx}{25-12x+4x^2} \;= \frac{7}{2}\int \frac{2\,dx}{(4)^2+(2x-3)^2} = \frac{7}{8}\ tan^{-1}(\tfrac{2x-3}{4}) \;+\; C$$