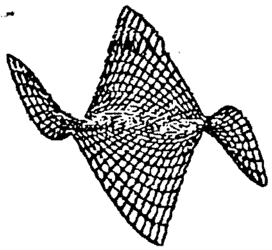


Mathematics Support Capsules

Limits: An Intuitive Approach

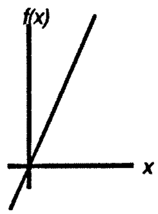
by Raymond W. Bacon

Limits are what makes calculus different from mathematics without calculus. The limit concept is the fundamental **new** idea in college mathematics. First year calculus covers two basic concepts: derivatives and integrals. Both are limits. It is important to get a good intuitive idea of what a limit is and what it isn't.

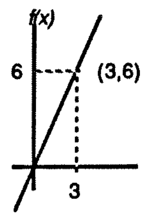
Consider the graph of the function $f(x) = 2x$.

This is an ordinary straight line.

Now consider the question: When x is **near** 3, what is $f(x)$ **near?**



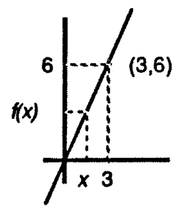
We know the answer to the following question: When x **is** 3, what **is** $f(x)$? That's easy: 6. That's the old stuff, nothing to do with limits. We can even plot (3,6) to represent this point:



But now look at an x **near** 3. What is $f(x)$ **near?** Again, the answer is 6. This seems obvious enough.

In fact, we now **say:** $\lim_{x \to 3} f(x) = 6$.

By this we mean, simply: When x is **near** 3, then $f(x)$ is **near** 6.



This is **NOT** the same as saying: when $x = 3$, then $f(x) = 6$. In this case, the two statements seem to be saying the same thing, but they are conceptually quite different.

It is necessary to belabor this difference because students so commonly don't get it the first time around.

This capsule was originally produced in 1980 as Mathematics Learning Module II through the Learning Skills Center-COSEP of Cornell University. This current (1987) form is produced by the Mathematics Support Center of Cornell University with minor revisions and format changes.

Again, one statement is:

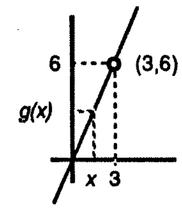
1) When x **is exactly** 3, then $f(x) = 2x$ **is exactly** 6.

The other statement is:

2) When x **is near** 3, then $f(x)$ **is near** 6.

Why is this difference important? For instance, if you know statement 2), then it seems like you know statement 1). **NOT TRUE!**

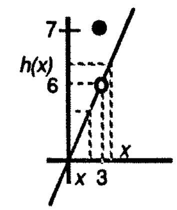
First, it might be impossible to know 1): Consider $g(x) = 2x\frac{(x-3)}{(x-3)}$. This function is not defined for $x = 3$ (because you can't divide by zero). It is the same function as $f(x) = 2x$, except there is a hole in it at the point (3,6). Thus, for this function, statement 1) cannot be made. But we can still make statement 2:



$\lim_{x \to 3} f(x) = 6$, because we can still see that when x is near 3, then $g(x)$ is near 6.

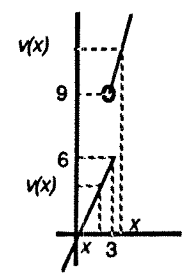
Second, statements 1) and 2) might be completely different. Consider $h(x) = \begin{cases} 2x & \text{if } x \neq 3 \\ 7 & \text{if } x = 3 \end{cases}$.

This is a perfectly acceptable function, but now statements 1) and 2) are completely different. For any x on either the left or right of 3, $h(x)$ is near 6, so $\lim_{x \to 3} h(x) = 6$. But when $x = 3$, $h(x) = 7$.



So, for a number a and some function $f(x)$, $\lim_{x \to a} f(x)$ and $f(a)$ are definitely not necessarily the same.

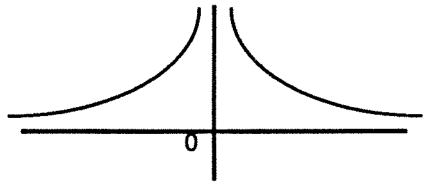
Furthermore, $\lim_{x \to a} f(x)$ doesn't even have to exist. Consider $v(x) = \begin{cases} 2x & \text{if } x \leq 3 \\ 3x & \text{if } x > 3 \end{cases}$. If x is near 3 **on the left,** then $v(x)$ is near 6. But if x is near 3 **on the right,** then $v(x)$ is near 9. This is a crucial point. If the left limit and the right limit are not the same, then we say that there is **no limit.** The notation for the left limit in this example is given by $\lim_{x \to 3^-} v(x) = 6$. For the right limit, $\lim_{x \to 3^+} v(x) = 9$.



Here's another example of a function without a limit for a certain *x*:

Let $f(x) = \frac{1}{x^2}$

Then $\lim_{x \to 0} \frac{1}{x^2} = ?$



When *x* is near 0, what is *f(x)* near? Well, suppose *x* gets closer and closer to 0:

If $x = 1/10$, then $f(x) = (10)^2$.

If $x = 1/100$, then $f(x) = (100)^2$.

If $x = 1/1000$, then $f(x) = (1000)^2$.

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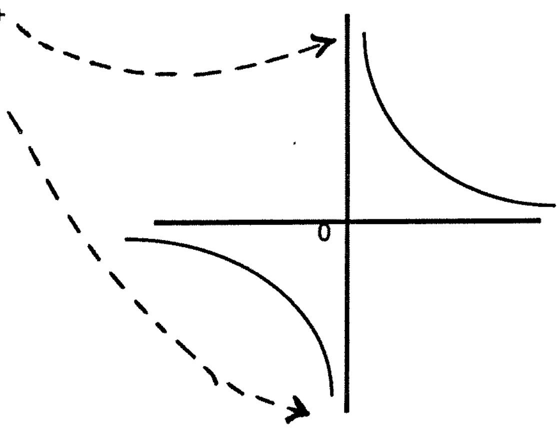
In other words, the closer *x* gets to 0, the bigger *f(x)* gets (that is, *f(x)* "goes to infinity".) If *f* doesn't get near any real number, then the limit does not exist. (∞ is not a number.)

Sometimes you see $\lim_{x \to 0} f(x) = \infty$, but in this capsule we will not use this notation. Instead we will say that the limit doesn't exist. You can express the idea of *f* "going to infinity" as *x* goes to 0 by using the notation $f \longrightarrow \infty$ as $x \longrightarrow 0$. For instance, in graphing $f(x) = \frac{1}{x}$, you might say:

$f \to +\infty$ as $x \to 0^+$

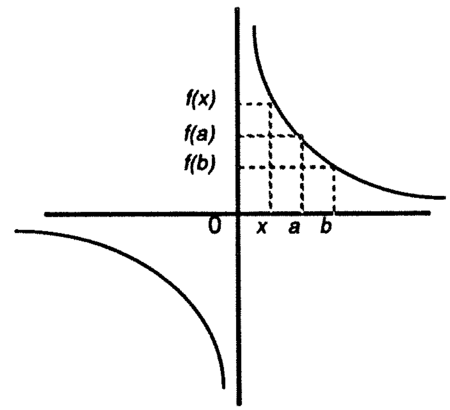
$f \to -\infty$ as $x \to 0^-$



We now have the notation: $\lim_{x \to a} (x) = L$, where L is a number. But we want to allow *a* to be either $+\infty$ or $-\infty$.

For instance, $\lim_{x \to +\infty} f(x) = ?$ reads "As *x* gets larger and larger ("goes to infinity"), what number does $\frac{1}{x}$ get near?"

You can see from the graph of $\frac{1}{x}$ (below) that as *x* gets farther to the right (larger), then *f(x)* gets closer and closer to 0. So $\lim_{x \to \infty} \frac{1}{x} = 0$.



Also, you can see that $\lim_{x \to -\infty} \frac{1}{x} = 0$. You should also be able to express the fact that $\lim_{x \to \infty} \frac{1}{x} = 0$ in words to yourself by saying something like: "If the denominator of a fraction gets larger and larger while the numerator stays fixed, then the fraction will get smaller and smaller until it approaches zero."

If 1) $\lim_{x \to a} (x)$ exists and 2) *f(a)* exists and 3) $\lim_{x \to a} f(x) = f(a)$, then the function is defined to be **continuous** at $x=a$. (That is, there is no hole or jump at $x=a$. You can draw the function at $x=a$ without picking up your pencil.)

For instance, $f(x) = 2x$ is continuous at $x=3$ because $\lim_{x \to 3} f(x) = 6 = f(3)$.

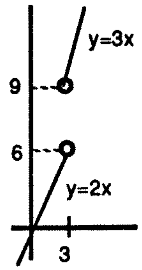
But $g(x) = 2x\frac{(x-3)}{(x-3)}$ is not continuous at $x=3$ because $\lim_{x \to 3} g(x) = 6 \neq g(3)$, because *g(3)* does not exist.

And $h(x) = \begin{cases} 2x \text{ if } x \neq 3 \\ 7 \text{ if } x=3 \end{cases}$ is not continuous at $x=3$ because $\lim_{x \to 3} h(x) \neq 7 = h(x)$.

And $v(x)=\begin{cases}2x \text{ if } x\le 3\\ 3x \text{ if } x>3\end{cases}$ is not continuous at x=3 because $\lim_{x\to 3} v(x) \neq 6 = f(3)$,

since $\lim_{x\to 3} v(x)$ doesn't exist.

Another possibility might be: $w(x)=\begin{cases}2x \text{ if } x<3\\ 3x \text{ if } x>3\end{cases}$ is not continuous because neither $\lim_{x\to 3} w(x)$ nor $w(3)$ exists.



All the discontinuous functions have holes or jumps in them. In other words, a function has a limit at a point only if it is continuous or **can be made continuous** at that point by "filling up the hole."

If the function is continuous, then you can find the limit just by "plugging in," since $\lim_{x\to a} f(x) = f(a)$.

For instance, $f(x) = 2x$ is an ordinary, continuous straight line. So $\lim_{x\to 3} 2x = 2(3) = 6$.

If the function has a hole in it which can be filled up by the point (a,y), then $\lim_{x\to a} f(x) = y$.

For instance, $g(x) = 2x\,\frac{(x-3)}{(x-3)}$ can be **made continuous** by simply **redefining it** to be: $g(x)=\begin{cases}2x\,\frac{(x-3)}{(x-3)} \text{ if } x\neq 3\\ 6 \text{ if } x=3\end{cases}$.

And $h(x)=\begin{cases}2x \text{ if } x\neq 3\\ 7 \text{ if } x=3\end{cases}$ can be made continuous by redefining it as in the following: $h(x)=\begin{cases}2x \text{ if } x\neq 3\\ 6 \text{ if } x=3\end{cases}$.

But if the function jumps at $x=a$ or "goes to infinity," or otherwise can't be made continuous, then the limit doesn't exist.

There is no way to make the jump in $v(x)=\begin{cases}2x \text{ if } x\le 3\\ 3x \text{ if } x>3\end{cases}$ continuous at x=3 by redefining the function at only one point.

Nor can $f(x) = \frac{1}{x^2}$ be made continuous at x=0.

Basically, you now know enough about limits to figure out how to solve the problems in a first year calculus course. You should realize, though, that there is more to limits, a "more rigorous" approach (using ε's and δ's to

express "nearness") which you need in order to "really understand" limits. Sometimes a course will require you to learn some simple ε-δ ideas. If you go on to "higher mathematics," you will want to learn how to do limits with ε's and δ's. But if you do, don't lose track of the common sense ideas of limits as so many students do.

Ask your instructor if there are going to be any ε-δ **proofs** of limits on your exams. Usually you can solve all the **problems** using only the following techniques and tricks:

[1.] **Factoring** EXAMPLE:

$$\lim_{x\to -1}\frac{x^2-1}{x+1} = \lim_{x\to -1}\frac{(x-1)(x+1)}{(x+1)} = -2.$$

In other words, we locate the hole in the function and cancel out the factors involved, yielding a continuous function (in this case $f(x)=x-1$) which is the same as the given function except for the hole. We can now simply plug in $x=a$ to fill up the hole.

Technically we have redefined the function $f(x)=\begin{cases}x-1 \text{ if } x\neq 1\\ \text{undefined at } x=1\end{cases}$ to be the function $f(x)=\begin{cases}x-1 \text{ if } x\neq -1\\ -2 \text{ if } x=-1\end{cases}$. You get the -2 by plugging x=-1 into x-1.

This method works for quotients of polynomials if you can cancel the vanishing factor in the denominator with the same factor in the numerator. BUT TAKE WARNING: This is where lots of students get the idea that the limit at $x=a$ is the same thing as plugging in a to get $f(a)$, which is **WRONG** for other kinds of limits problems, as we have seen.

EXAMPLE:

$$\lim_{x\to 3}\frac{x^3+2x^2-9x-18}{x^4-81} = \lim_{x\to 3}\frac{(x^2-9)(x+2)}{(x^2-9)(x^2+9)} = \lim_{x\to 3}\frac{x+2}{x^2+9} = \frac{5}{18}.$$

EXAMPLE: Redefine $f(x) = \frac{(x+1)(x-2)(x+2)}{(x-2)(x+2)}$ so that it will be continuous at any real number.

There are holes at x=2 and x=-2.

So let $f(x)=\begin{cases}x+1 \text{ if } x\neq 2,-2\\ 3 \text{ if } x=2\\ -1 \text{ if } x=-2\end{cases}$.

[2.] **Multiplying by Conjugates** (You need 2 and 3 later for calculating derivatives from the definition.)

$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \lim_{h\to 0}\frac{(x+h)-(x)}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

3. Combining Fractions

$$\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \lim_{h\to 0}\frac{\frac{x-(x+h)}{(x+h)x}}{h} = \lim_{h\to 0}\frac{-\cancel{h}}{\cancel{h}(x+h)x} = -\frac{1}{x^2}$$

4. $x\to\infty$, $x\to-\infty$

Remember $\lim_{x\to\pm\infty}\frac{1}{x} = 0$, $\lim_{x\to\pm\infty}\frac{1}{x^2} = 0$, etc.

For quotients of polynomials, divide top and bottom by a power of x (usually the highest power in top or bottom). For example:

$$\lim_{x\to\infty}\frac{x}{x+1} = \lim_{x\to\infty}\frac{1}{1+1/x} = 1$$

[NOTE: Technically you're using a theorem here: $\lim\left(\frac{f}{g+h}\right) = \frac{\lim f}{\lim g+\lim h}$. Some courses require you to learn these theorems exactly and use them to justify your solutions.]

Here's an example dividing the numerator and denominator by x^6:

$$\lim_{x\to\infty}\frac{4x^6+3x}{3x^6+x^5+5} = \lim_{x\to\infty}\frac{4+3/x^5}{3+1/x+5/x^6} = \frac{4}{3}$$

Here's an example dividing by x^3:

$$\lim_{x\to\infty}\frac{x^3}{x^2+1} = \lim_{x\to\infty}\frac{1}{1/x+1/x^3} = \frac{1}{\text{smaller \& smaller}},$$ therefore **no limit**.

Here's the same example, dividing by x^2:

$$\lim_{x\to\infty}\frac{x^3}{x^2+1} = \lim_{x\to\infty}\frac{x}{1+1/x^2} = x,$$ therefore **no limit**.

5. Absolute Values (Divide into cases, check right side & left side limits)

Remember definition

$$|\alpha| = \begin{cases}\alpha \text{ if } \alpha>0, & \text{case 1}\\ 0 \text{ if } \alpha=0 & \\ -\alpha \text{ if } \alpha<0, & \text{case 2}\end{cases}$$

Now, consider $\lim_{x\to 3}\frac{|x-3|}{x-3}$. Using the above definition:

$$|x-3| = \begin{cases}x-3 \text{ if } x-3>0, (\text{i.e. if } x>3), & \text{case 1}\\ 3-x \text{ if } x-3<0, (\text{i.e. if } x<3), & \text{case 2}\end{cases}$$

So, (case 1) $\lim_{x\to 3^+}\frac{|x-3|}{x-3} = \lim_{x\to 3^+}\frac{x-3}{x-3} = 1$

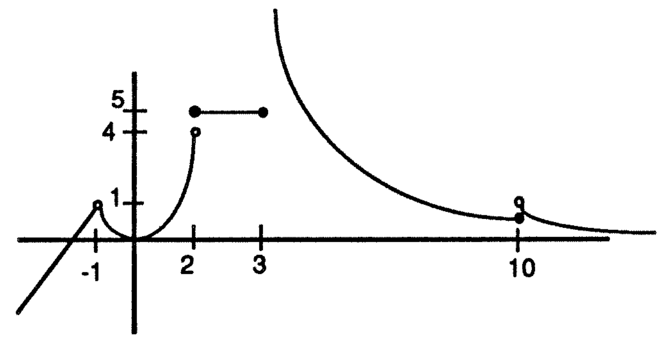
And, (case 2) $\lim_{x\to 3^-}\frac{|x-3|}{x-3} = \lim_{x\to 3^-}\frac{3-x}{x-3} = -1$

THEREFORE NO LIMIT! (The left and right side limits have to be the **same** for the limit to exist.)

6. Odd-ball functions (Draw and see what happens!)

$$\text{Consider } f(x) = \begin{cases}2x+3 \text{ if } x<-1\\ x^2 \text{ if } -1<x<2\\ 5 \text{ if } 2\le x\le 3\\ \frac{1}{x-3} \text{ if } 3<x\le 10\\ \frac{1}{x-4} \text{ if } x>10\end{cases}$$

Where is f continuous? discontinuous? For what a's does $\lim_{x\to a} f(x)$ exist?



Answer: f is continuous for all reals except -1,2,3, & 10, so $\lim_{x\to a} f(x) = f(a)$ for all reals except these four.

$\lim_{x\to -1} f(x) = 1.$

$\lim_{x\to 2} f(x)$ doesn't exist since $\lim_{x\to 2^-} f(x) = 4$ and $\lim_{x\to 2^+} f(x) = 5$.

$\lim_{x\to 3} f(x)$ doesn't exist since $\lim_{x\to 3^-} f(x) = 5$, but as $x\to 3^+$, $f\to+\infty$.

$\lim_{x\to 10} f(x)$ doesn't exist since $\lim_{x\to 10^-} f(x) = 1/7$ and $\lim_{x\to 10^+} f(x) = 1/6$.

$\lim_{x\to\infty} f(x) = 0.$

$\lim_{x\to-\infty} f(x)$ doesn't exist since as $x\to-\infty$, $f\to-\infty$.

7. **Memorize:** $\lim_{x\to 0}\frac{\sin x}{x} = 1$ and $\lim_{x\to 0}\cos x = 1$.

$$\left(\lim_{x\to 0}\frac{x}{\sin x} = 1 \text{ also}\right)$$

The key to using this fact lies in realizing what it says. That is, if something (say α) approaches 0 and the problem has a $\sin\alpha$ in it, then if you can get the same thing, α, either below it $\left(\frac{\sin\alpha}{\alpha}\right)$ or above it $\left(\frac{\alpha}{\sin\alpha}\right)$, then the quotient will approach 1.

So consider $\lim_{x\to 0}\frac{\sin 2x}{x}$ and let $\alpha = 2x$. Then $\alpha\to 0$ as $x\to 0$, so multiply both the top and bottom by 2:

Then $\lim_{x\to 0}\frac{\sin 2x}{x} = \lim_{x\to 0} 2\left(\frac{\sin 2x}{2x}\right) = 2.$

Or consider $\lim_{x\to 0}\frac{3x}{\tan 5x} = \lim_{x\to 0} 3x\frac{\cos 5x}{\sin 5x}$. Now let $\alpha = 5x$.

Then $\alpha\to 0$ as $x\to 0$, so multiply the top and bottom by 5:

$$= \lim_{x\to 0}\frac{3}{5}(\cos 5x)\frac{5x}{\sin 5x} = \frac{3}{5}$$

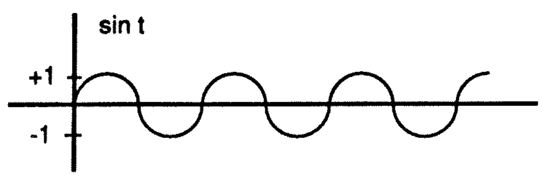
8. **Common Sense**

$\lim_{x\to\pm\infty} x$ doesn't exist $\qquad \lim_{x\to\pm\infty} 5 = 5$

$\lim_{x\to 5} 5 = 5$ (When x gets near 5, 5 not only gets near, but is, 5)

$\lim_{x\to 0}\frac{1}{\sqrt{x}}$ doesn't exist $\qquad \lim_{t\to 1}\frac{\sqrt{t^{20}+t^7}}{1+t} = \frac{\sqrt{2}}{2}$ (just plug in!)

$\lim_{t\to\infty}\sin t$ doesn't exist. The function oscillates between -1 and 1, and doesn't get **near** any number.



$\lim_{x\to 0} x\sin x = (0)(0) = 0.$

$\lim_{x\to\infty} x\sin x$ doesn't exist since $\sin x$ oscillates between -1 and 1.

9. **L'Hospital's Rule:** You have to know how to calculate derivatives in order to use this. If you don't, come back to it later in the term.

Use on quotients of functions: $\frac{f(x)}{g(x)}$. If, as $x\to a$, $\frac{f(x)}{g(x)}\to\frac{0}{0}$ or $\frac{\infty}{\infty}$, then--

--look at $\frac{f'(x)}{g'(x)}$. If $\frac{f'(x)}{g'(x)}\to L$, then $\frac{f(x)}{g(x)}\to L$.

EXAMPLE: $\lim_{x\to 3}\frac{x-3}{x-3} = \lim_{x\to 3}\frac{1}{1} = 1.$

$\lim_{x\to\infty}\frac{x+3}{17-x} = \lim_{x\to\infty}\frac{1}{-1} = -1.$

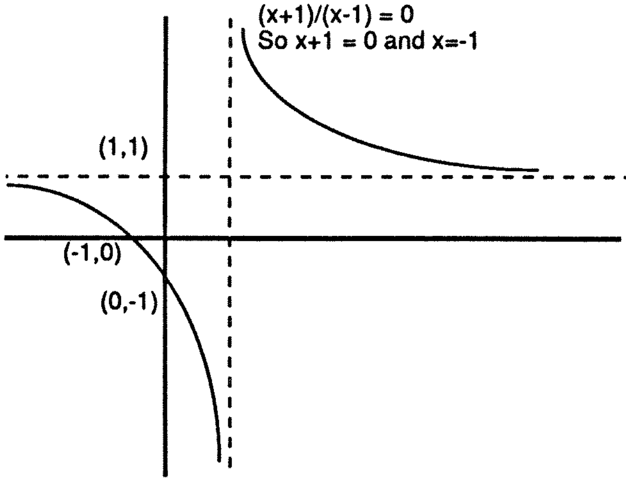
$\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{1/x}{1} = 0.$

$\lim_{x\to\infty}\frac{x^2}{e^x} = \lim_{x\to\infty}\frac{2x}{e^x} = \lim_{x\to\infty}\frac{2}{e^x} = 0.$

You should be able to use limits, even ones that don't exist, to help you **graph** functions.

$f(x) = \frac{x+1}{x-1}$

x cannot be +1, so there will be a vertical asymptote at $x = +1$. At $x = 0$, $f(x) = \frac{0+1}{0-1} = -1$. (Plug in an easy point just to get started.)



Establish $\lim_{x\to\infty}\frac{x+1}{x-1} = 1$ and $\lim_{x\to-\infty}\frac{x+1}{x-1} = 1$ to get horizontal asymptote.

Check $\lim_{x\to 1^-}\frac{x+1}{x-1}$; it doesn't exist, but as $x \to 1^-$, $f \to -\infty$. Finally, check

$\lim_{x\to 1^+}\frac{x+1}{x-1}$; as $x \to 1+$, $f \to +\infty$.

EXERCISES (WITH SOLUTIONS)
to accompany
LSC Mathematics Learning Module II
Limits: An Intuitive Approach
compiled by the Mathematics Support Center (8/81)

When using limits to graph, you may want to refer to the Math Support Center's capsules on graphing, available at the Mathematics Support Center.

1. graph $f(x) = \frac{x}{|x|}$.

$\lim_{x \to 0^-} f(x) = ?$ (i.e. what is the limit as x approaches 0 from the left?)

$\lim_{x \to 0^+} f(x) = ?$ (i.e. what is the limit as x approaches 0 from the right?)

Does the limit for $x \to 0$ exist?

2. Let $f(x) = 4x^3$. Then find the following limits (if the limit approaches ∞ or $-\infty$, indicate that the limit doesn't exist).

a. $\lim_{x \to 0^+} f(x) =$

b. $\lim_{x \to 0^-} f(x) =$

c. $\lim_{x \to \infty} f(x) =$

d. $\lim_{x \to -\infty} f(x) =$

e. graph $f(x)$.

3. Find the limit, as $x \to 2$, of $\frac{x^2+x-6}{x-2}$.

4. Find the limit,as $x \to 2$, of $\frac{x^2-1}{x^3-4}$.

5. Redefine $f(x) = \frac{x^3-x+3x^2-3}{x-1}$ so that it will be continuous at any real number.

6. Find the limit, as $x \to \pm\infty$, of $f(x) = \frac{(x+1)(x^2-2)}{x^2+1}$. What is the limit as x approaches 0?

7. Find the limit, as $x \to \pm\infty$, of $f(x) = \frac{17x^3+214x^2+56x}{4x^3+11}$.

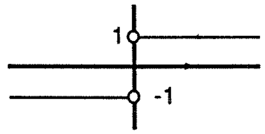
8. Find the limit, as $x \to 1$, of $f(x) = \frac{|x-1|}{x-1}$.

9. $\lim_{x \to 0} \frac{\sin x - x}{x^3} =$

10. $\lim_{t \to \infty} \frac{t^2+t}{2t^2+1} =$

Solutions
to
Limits Exercises

1. If $f(x) = \frac{x}{|x|}$, then:



$\lim_{x \to 0^-} f(x) = -1$, and

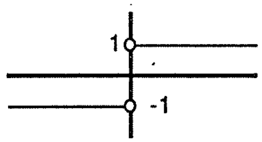
$\lim_{x \to 0^+} f(x) = 1$

The limit for $x \to 0$ does not exist, because the left limit and the right limit are not the same. There is no limit.

2. If $f(x) = 4x^3$, then: a. $\lim_{x \to 0^+} f(x) = 0$. b. $\lim_{x \to 0^-} f(x) = 0$.

c. $\lim_{x \to \infty} f(x)$ doesn't exist d. $\lim_{x \to -\infty} f(x)$ doesn't exist.

e.



3. $\frac{x^2+x-6}{x-2}$ factors into $\frac{(x+3)(x-2)}{(x-2)} = x+3$.
$(x+3)$ is continuous and therefore the limit of $x+3$ as $x \to 2$ can be found by putting $x=2$ into $x+3$ to get our answer of (+5).

4. In this case the denominator doesn't vanish at $x=2$. Therefore the limit can be obtained by replacing x by 2 directly into $\frac{x^2-1}{x^3-4}$. So $\frac{(2)^2-1}{(2)^3-4} = 3/4$.

5. As it stands now, $f(x)$ is continuous except at $x = 1$. Now $f(x) = \frac{(x^3-x+3x^2-3)}{x-1} = \frac{(x-1)(x+1)(x+3)}{(x-1)}$, thus for $x \neq 1$ (can't divide by 0) $f(x) = (x+1)(x+3)$. So for $x=1$, we <u>want</u> $f(x)$ to have the value $(x+1)(x+3)$, or $f(1) = 8$.

So redefine $f(x) = \begin{cases} (x+1)(x+3) & \text{if } x \neq 1 \\ 8 & \text{if } x = 1 \end{cases}$

6. $f(x) = \frac{(x+1)(x^2-2)}{x^2+1} = \frac{x^3+x^2-2x-2}{x^2+1}$
(Then divide top & bottom by x^3, the largest power of x)

So $\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{\frac{x^3}{x^3} + \frac{x^2}{x^3} - \frac{2x}{x^3} - \frac{2}{x^3}}{\frac{x^2}{x^3} + \frac{1}{x^3}} = \lim_{x \to \pm\infty} \frac{1 + \frac{1}{x} - \frac{2}{x^2} - \frac{2}{x^3}}{\frac{1}{x} + \frac{1}{x^3}}$

As $x \to \pm\infty$, the denominator $\to 0$: so <u>the limit doesn't exist</u>.
When $x \to 0$, then the denominator doesn't vanish in $f(x) = \frac{(x+1)(x^2-2)}{x^2+1}$,
so we simply replace x by 0: $\frac{0-2}{0+1} = \boxed{-2}$.

7. Dividing top and bottom by x^3 gives $\frac{17 + 214/x + 56/x^2}{4 + 11/x^3} = 17/4$.

8. $\lim_{x \to 1^+} \frac{x-1}{x-1} = 1$ $\quad \lim_{x \to 1^-} \frac{1-x}{x-1} = -1$, so no limit.

9. $\lim_{x \to 0} \frac{\sin x - x}{x^3} = \lim_{x \to 0} \frac{\cos x - 1}{3x^2} = \lim_{x \to 0} \frac{-\sin x}{6x} = \lim_{x \to 0} \frac{-\cos x}{6} = -1/6$.

10. This assumes the form $\frac{\infty}{\infty}$ as $t \to \infty$:

a) $\lim_{t \to \infty} \frac{t^2+t}{2t^2+1} = \lim_{t \to \infty} \frac{2t+1}{4t} = \lim_{t \to \infty} \frac{2}{4} = 1/2$.

OR b) $\lim_{t \to \infty} \frac{t^2+t}{2t^2+1} = \lim_{t \to \infty} \frac{\frac{t^2}{t^2} + \frac{t}{t^2}}{2\frac{t^2}{t^2} + \frac{1}{t^2}} = \lim_{t \to \infty} \frac{1 + \frac{1}{t}}{2 + \frac{1}{t^2}} = 1/2$.