Mathematics Support Capsules

LOGARITHMS

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LOGARITHMS

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I. LOGARITHMS BASE 10

$\boxed{\log_{10} X}$ is the number to which you must raise 10 to get X.

$\boxed{\text{so: } \log_{10} X = Y \quad \text{means} \quad 10^Y = X}$

Y is called the logarithm of X, base 10. Notice that Y is an exponent.

Think of it this way: $\log_{10} X = Y$ following the arrows we get $10^Y = X$

examples:

1. $\log_{10} 100 = 2$ since $10^2 = 100$
2. $\log_{10} .001 = -3$ since $10^{-3} = .001$

in example 1, the logarithm is $\boxed{2}$.

in example 2, it's $\boxed{-3}$.

Exercises:

	SOLVE FOR x:	
1) $\log_{10} 1000 = ?$	6) $\log_{10} x = 5$	x = ?
2) $\log_{10} .01 = ?$	7) $\log_{10} x = 1/2$	x = ?
3) $\log_{10} 10 = ?$	8) $\log_{10} x = 0$	x = ?
4) $\log_{10} 1 = ?$	9) $\log_{10} x = -1$	x = ?
5) $\log_{10} 10^{25} = ?$	10) $\log_6 3x = 1$	x = ?

Answers:

1) 3	6) 10^5 or 100,000
2) -2	7) $10^{1/2}$ or $\sqrt{10}$
3) 1	8) 10^0 or 1
4) 0	9) 10^{-1} or 1/10
5) 25	10) 2

II. LOGARITHMS BASE b

$\boxed{\text{Log}_b\ X}$ is the number to which you must raise b to get X. b can be any positive real number except 1.

$\boxed{\text{so: } \text{Log}_b\ X = Y \text{ means } b^Y = X}$

Y is called the logarithm of X, base b. Notice that Y is an exponent.

Think of it this way: $\text{Log}_b\ X = Y$ follow the arrows to get: $b^Y = X$

examples:

1. $\text{Log}_2\ 8 = 3$ since $2^3 = 8$
2. $\text{Log}_3\ (1/9) = -2$ since $3^{-2} = 1/9$
3. $\text{Log}_4\ 2 = 1/2$ since $4^{1/2} = \sqrt{4} = 2$

in example 1, the logarithm is $\boxed{3}$.

in example 2, it's $\boxed{-2}$.

in example 3, it's $\boxed{1/2}$.

Exercises:

1) $\text{Log}_5\ 25 = ?$
2) $\text{Log}_9\ 3 = ?$
3) $\text{Log}_2\ (1/16) = ?$
4) $\text{Log}_{1/2}\ (1/4) = ?$
5) $\text{Log}_{1/2}\ 4 = ?$

SOLVE FOR x:

6) $\text{Log}_7\ x = -2$
7) $\text{Log}_{1/2}\ x = 3$
8) $\text{Log}_{1/4}\ x = -2$

Answers:

1) 2
2) 1/2
3) -4
4) 2
5) -2
6) 7^{-2} or 1/49
7) $(\frac{1}{2})^3$ or 1/8
8) $(\frac{1}{4})^{-2}$ or 16

III. PROPERTIES OF LOGARITHMS

NOTE: If you see $\boxed{\text{Log } X}$ with no base b, it means the base is 10.

So: $\boxed{\text{Log } X = \text{Log}_{10}\ X}$.

Recall the properties of exponents:

1. $x^m x^n = x^{m+n}$
2. $x^m / x^n = x^{m-n}$
3. $(x^m)^n = x^{mn}$
4. $x^0 = 1$
5. $x^1 = x$

For explanation of these properties, see capsule on EXPONENTS.

Since the Log of a number is the exponent of the number, we get the following properties of logarithms:

1. $\text{Log}_b\ (XY) = \text{Log}_b\ X + \text{Log}_b\ Y$
2. $\text{Log}_b\ (X/Y) = \text{Log}_b\ X - \text{Log}_b Y$
3. $\text{Log}_b\ X^n = n\ \text{Log}_b\ X$
4. $\text{Log}_b\ 1 = 0$ (since $b^0 = 1$)
5. $\text{Log}_b\ b = 1$ (since $b^1 = b$)

NOTE: YOU CANNOT TAKE THE LOG OF A NEGATIVE NUMBER OR OF ZERO:

example: $\text{Log}_{10}\ (-5)$ IS NOT DEFINED.

EXAMPLES

1) Simplify as much as possible: $\text{Log}_b(\frac{X^2Y}{z^3})$

$\text{Log}(\frac{X^2Y}{z^3}) = \text{Log}_b(X^2Y) - \text{Log}_b(z^3)$ using property 2

$= \text{Log}_b(x^2) + \text{Log}_b Y - \text{Log}_b(z^3)$ using property 1

$= \boxed{2\ \text{Log}_b\ x + \text{Log}_b Y - 3\ \text{Log}_b z}$ using property 3

Examples cont'd.

2) <u>Put into one logarithm:</u> $\log_5 9 + 3\log_5 2 - 3\log_5 3$

[note that the base must be the same in each term].

$\log_5 9 + 3\log_5 2 - 3\log_5 3$

$= \log_5 9 + \log_5 2^3 - \log_5 3^3$ using property 3

$= \log_5 (9 \cdot 2^3) - \log_5 3^3$ using property 1

$= \log_5 \left(\frac{9 \cdot 2^3}{3^3}\right)$ using property 2

$= \log_5\left(\frac{72}{27}\right) = \log_5(8/3)$ multiplying together

EXERCISES

I. <u>Simplify</u>

a) $\text{Log}_2\,(2x)$

b) $\text{Log}_b\,\left(\frac{xy}{z}\right)^{1/2}$

c) $\text{Log}_{10}\,(\sqrt[3]{x}\ \sqrt{y^3}\,)$

d) $\text{Log}\,(x^2-1)$

II. <u>Put into a single Logarithm</u>

a) $\frac{1}{2}\,(\text{Log}_b X + \text{Log}_b Y)$

b) $4\,\text{Log}\,3 - 2\,\text{Log}\,3$

c) $2\,\text{Log}\,3 - \frac{1}{2}\,\text{Log}\,9$

d) $10(\text{Log}_8 X - \text{Log}_8 Y - \text{Log}_8 Z)$

<u>Answers</u>

I.
a) $1 + \text{Log}_2 X$

b) $\frac{1}{2}\text{Log}_b X + \frac{1}{2}\text{Log}_b Y - \frac{1}{2}\text{Log}_b Z$

c) $\frac{1}{3}\text{Log}_{10} X + \frac{3}{2}\text{Log}_{10} Y$

d) $\text{Log}(x+1) + \text{Log}(x-1)$

II.
a) $\text{Log}_b\,(XY)^{1/2}$

b) Log 9

c) Log 3

d) $\text{Log}_8(X/YZ)^{10}$

IV. MORE PROPERTIES OF LOGARITHMS AND CHANGE OF BASE

A. If $X = Y$, then $\text{Log}_b X = \text{Log}_b Y$

B. If $\text{Log}_b X = \text{Log}_b Y$, then $X = Y$

C. $\text{Log}_a X = \dfrac{\text{Log}_b X}{\text{Log}_b a}$ (See an algebra or calculus text for explanation of this formula to change from $\boxed{\text{Log}_b X}$ to $\boxed{\text{Log}_a X}$.)

D. $b^{\text{Log}_b X} = X$ (since $\text{Log}_b X$ is what you must raise b to, to get X).

E. $\text{Log}_b b^m = m$ (since $\text{Log}_b b^m = m\,\text{Log}_b b = m \cdot 1 = m$)
(prop 3, prop 5)

We'll use these properties in the next section on solving equations.

V. SOLVING LOGARITHMIC EQUATIONS

Examples:

1) Solve for x: $5^x = 6$

we can solve two different ways:

1) NOTICE $5^x = 6$ means $\log_5 6 = x$ using logarithmic notation.

2) OR you can just take the log of both sides, using property A: Since $5^x = 6$, we know $\log_b 5^x = \log_b 6$ (we can use any "b" we want)

NOW:

$\log_b 5^x = \log_b 6$, and

$x \log_b 5 = \log_b 6$ using property 3

so: $x = \dfrac{\log_b 6}{\log_b 5}$

(NOTE: If we pick b=6, then $x = \dfrac{\log_6 6}{\log_6 5} = \dfrac{1}{\log_6 5}$ (since $\log_6 6 = 1$)

If we pick b=5, then $x = \dfrac{\log_5 6}{\log_5 5} = \dfrac{\log_5 6}{1}$ (since $\log_5 5 = 1$))

2) Solve for x: $2^{x-1} = 5^x$

1: take logs of both sides using b=2 so:

$\log_2 2^{x-1} = \log_2 5^x$ (using A)

2: $(x-1) \log_2 2 = x \log_2 5$ using property 3

3: $(x-1) = x \log_2 5$ using property 5

4: $x - x \log_2 5 = 1$ get all the x-terms on one side.

5: $x(1-\log_2 5) = 1$ factor out the x.

6: $\boxed{x = 1/(1-\log_2 5)}$ divide

NOTE: YOU CAN ALSO USE b = 5 here. Try it!

Examples cont'd.

3. Simplify: $10^{2 \log 5x}$ (No b here since base = 10).

We know that $10^{\log_{10} a} = a$ by property D; so we'd like to get $10^{2 \log 5x}$ in this form.

Well: $2 \log 5x = \log(5x)^2$ by 3

So: $10^{2 \log 5x}$

$= 10^{\log(5x)^2} = 10^{\log \boxed{25x^2}} = 25x^2$ by D.

4. Solve for x: $\log_8 4x + \log_8 2 = 2$

$\log_8 4x + \log_8 2 = 2$

$\log_8 (4x \cdot 2) = 2$ using property 1

convert to exp. form:

$\log_8 (8x) = 2$

so $8^2 = 8x$

so $\boxed{x = 8}$

Strategy: Aim for a single logarithm on the left so you can convert to exponential form.

EXERCISES

I. Solve for X:

a) $7^x = 15$ b) $5^{x-1} + 5 = 30$

c) $3^x = 4^{x-1}$ d) $3^{2x} = 9^{3x-4}$

e) $\mathrm{Log}_2\, x - \mathrm{Log}_2(x+4) = -5$ f) $\mathrm{Log}(\mathrm{Log}\, x) = 3$

g) $\mathrm{Log}_4\,(x+3) + \mathrm{Log}_4\,(x-3) = 2$ h) $\mathrm{Log}_5 \sqrt{x^2+9} = 1$

II. Simplify:

a) $3^{5\,\mathrm{Log}_3\, 2x}$ b) $2^{3\,\mathrm{Log}_2\, x - 8\,\mathrm{Log}_2\, Y}$

c) $b^{\log_b x + \log_b(\frac{1}{x})}$ d) $10^{2+\mathrm{Log}\, x}$

<u>Answers:</u> I. (a) $x = \mathrm{Log}_7\, 15$ or $\frac{\mathrm{Log}_b 15}{\mathrm{Log}_b 7}$ (for any b)

(b) x=3 (c) $x = \frac{\mathrm{Log}_b 4}{\mathrm{Log}_b 4 - \mathrm{Log}_b 3}$ (for any b)

(d) $x = \frac{4\,\mathrm{Log}_b 9}{3\,\mathrm{Log}_b 9 - 2\,\mathrm{Log}_b 3}$ or 2 (if you simplify)

(e) x = 4/31 (f) $x = 10^{1000}$

(g) x = 5 (h) x=4 or x = -4

II. a) $32x^5$ b) x^3/Y^8

c) 1 d) 100x

VI. GRAPHING EXPONENTIAL AND LOGARITHMIC FUNCTIONS

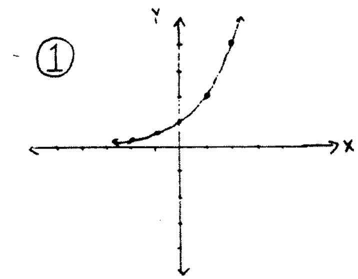
1) $Y = 2^X$

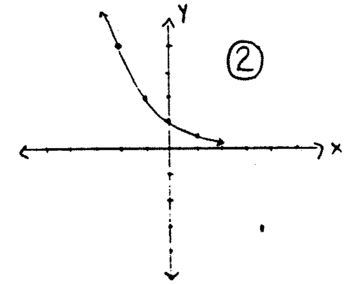
Plot points, picking points which are easy to graph. Then connect the points. 5 points are usually good enough.

X	-2	-1	0	1	2
Y	1/4	1/2	1	2	4

(2) $Y = (1/2)^X$

X	-2	-1	0	1	2
Y	4	2	1	1/2	1/4





Notice that these 2 graphs are symmetric or "mirror images" of each other across the Y-axis. Also notice that Y is <u>always</u> <u>positive</u>.

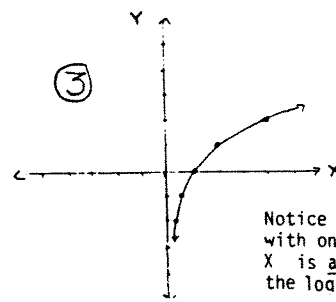
3) $Y = \mathrm{Log}_2\, X$

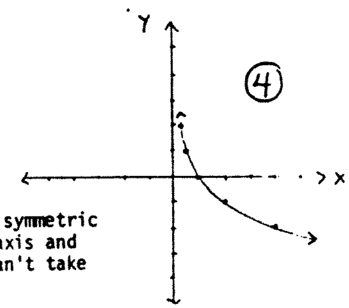
Pick values for X that are easy to compute the log of

X	1/2	1/4	1	2	4
Y	-1	-2	0	1	2

(4) $Y = \mathrm{Log}_{1/2}\, X$

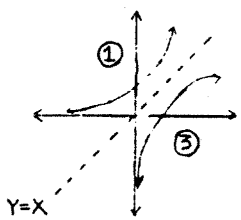
X	1/2	1/4	1	2	4
Y	1	2	0	-1	-2



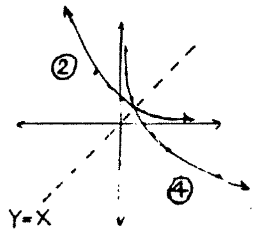


Notice that these 2 graphs are symmetric with one another across the x-axis and X is <u>always positive</u>. (You can't take the log of a positive number).

Also: Notice that 1 and 3 and 2 and 4 are symmetric to each other across the line $Y = X$



1 and 3 are called inverses of one another



2 and 4 are called inverses of one another

So $\boxed{Y=2^X}$ and $\boxed{Y = \text{Log}_2 X}$ are inverses of each other, also $\boxed{Y = (1/2)^X}$ and $\boxed{Y = \text{Log}_{1/2} X}$ are inverses of each other.

> The logarithm function is just the inverse of the exponential function

For more about inverse functions, see William K. Smith, Inverse Functions (Macmillan) - it's a small paperback, and a copy has been placed in the MSC Browsing Library.

EXERCISE:

Graph: 1) $\text{Log } Y = 10^X$ and 2) $Y = \text{Log}_{10} x$ --- by plotting at least 4 points.

VII. SUMMARY, FOR NATURAL LOGARITHMS

NOTE: If you haven't seen logarithms base e, you will see them in calculus.

PROPERTIES OF $\ell n\ x$ and e^x

1. $\ell n(AB) = \ell n\ A + \ell n\ B$
2. $\ell n(A/B) = \ell n\ A - \ell n\ B$
3. $\ell n(A^B) = B\ \ell n\ A$
4. $\ell n\ e^x = x$
5. $e^{\ell n\ x} = x$
6. $e^{A+B} = e^A e^B$
7. $e^{A-B} = e^A/e^B$
8. $(e^A)^B = e^{AB}$

4) SOLVE FOR x:

$\ell n\ x + \ell n(x-4) = \ell n\ 5$

$\ell n[x(x-4)] = \ell n\ 5$ by 1

so $e^{\ell n\ x(x-4)]} = e^{\ell n\ 5}$

and $[x(x-4)] = 5$ by 5

$x^2 - 4x = 5$

$x^2 - 4x - 5 = 0$

$(x-5)(x+1) = 0$

$\boxed{\text{So } x = 5,\ x = -1}$

EXAMPLES:

1) SIMPLIFY: $e^{4\ \ell n\ x\ -\ 5\ \ell n\ Y} = \dfrac{e^{4\ \ell n\ x}}{e^{5\ \ell n\ y}}$ by 7

$= \dfrac{e^{\ell n\ x^4}}{e^{\ell n\ y^5}}$ by 3

$= \dfrac{x^4}{y^5}$ by 5

2) SIMPLIFY: $\ell n(e^{3x}\sqrt{x}) = \ell n\ e^{3x} + \ell n\sqrt{x}$ by 1

$= 3x + \ell n\ x^{1/2}$ by 5

$= 3x + \frac{1}{2}\ \ell n\ x$ by 3

3) $\ell n(x+y)$ does not reduce, it can't be simplified.