Summer Math Institute Analysis

Homework Solutions

Summer 2008

1 June 10, 2008

Exercise 1.1.

Let $y \in \mathbb{R}$. Then solving $x^2 - 6x + 5 = y$ using the quadratic formula we have

$$\begin{aligned} x^2 - 6x + (5 - y) &= 0 \\ x &= \frac{6 \pm \sqrt{36 - 4 \cdot 1 \cdot (5 - y)}}{2 \cdot 1} \\ x &= \frac{6 \pm \sqrt{16 + 4y}}{2} \end{aligned}$$

For $y \in (-\infty, -4)$, $16 + 4y < 0$ so we have no real numbers that map to y under f. So f maps $\mathbb{R}$ into $[-4, \infty)$. For $y \in [-4, \infty)$, $16 + 4y \geq 0$ so we have a real number that maps to y under f (in fact two real numbers when $y > -4$. Therefore f maps $\mathbb{R}$ onto $[-4, \infty)$, as desired.

Exercise 1.2.

(a) Let $x, y \in A$ such that $(g \circ f)(x) = (g \circ f)(y)$. By the definition of composition we have $g(f(x)) = g(f(y))$. $f(x)$ and $f(y)$ are elements of B, so since g is one-to-one $g(f(x)) = g(f(y))$ implies that $f(x) = f(y)$. Then since f is one-to-one we have $x = y$. Therefore $g \circ f$ is one-to-one, as desired.

(b) Let $c \in C$. Since g is onto C there exists $b \in B$ such that $g(b) = c$. Since f is onto B there exists $a \in A$ such that $f(a) = b$. Then $(g \circ f)(a) = g(f(a)) = g(b) = c$. Therefore $g \circ f$ is onto.

(c) Let $x, y \in A$ such that $f(x) = f(y)$. Then $(g \circ f)(x) = g(f(x)) = g(f(y)) = (g \circ f)(y)$. Since $g \circ f$ is one-to-one, this implies that $x = y$, so f is also one-to-one.

Note that if f is not onto B, then g need not be one-to-one in this case!

(d) Let $c \in C$. Then since $g \circ f$ is onto C there exists $a \in A$ such that $(g \circ f)(a) = c$. So we have $f(a) \in B$ such that $g(f(a)) = (g \circ f)(a) = c$. Therefore g is onto C.

Note that f need not be onto.

Exercise 1.3.

(a) Let $f : [-6, 10] \to \mathbb{R}$ be given by $f(x) = \frac{3}{16}(x + 6) + 1$. Since f is an increasing function it is one-to-one. For $x \geq -6$, $f(x) \geq 1$ and for $x \leq 10$, $f(x) \leq 4$, so $f(x)$ maps into $[1, 4]$. For $y \in [1, 4]$, $\frac{16}{3}(y-1)-6 \in [-6, 10]$ and $f(\frac{16}{3}(y-1)-6) = y$, so f is onto $[1, 4]$. So $f : [-6, 10] \to [1, 4]$ is a bijection which implies $[-6, 10] \sim [1, 4]$.

(b) Let $g : (-\infty, 3) \to \mathbb{R}$ be given by $g(x) = -x + 4$. Then g is decreasing hence one-to-one. For $x < 3$, $g(x) > 1$, so g maps into $(1, \infty)$ For $y \in (1, \infty)$, $-y + 4 \in (-\infty, 3)$ and $g(-y + 4) = y$, so g is onto $(1, \infty)$. So $g : (-\infty, 3) \to (1, \infty)$ is a bijection which implies $(-\infty, 3) \sim (1, \infty)$.

(c) Let $h : (-\infty, 1) \to \mathbb{R}$ be given by $h(x) = \frac{1}{-x+2} + 1$ (h can be thought of as mapping the interval $(-\infty, 1)$ to $(-1, \infty)$, then $(1, \infty)$, then $(0, 1)$ and then $(1, 2)$). Then by the same arguments as the previous parts h can be shown to be a bijection from $(-\infty, 1)$ to $(1, 2)$, so $(-\infty, 1) \sim (1, 2)$.

(d) Define $f : \mathbb{J} \cup \{0\} \to \mathbb{Z}$ by

$$f(n) = \begin{cases} \frac{n}{2} & n \text{ even} \\ -\frac{n+1}{2} & n \text{ odd} \end{cases}$$

To see that f is one-to-one, assume that $m, n \in \mathbb{J} \cup \{0\}$ and $f(m) = f(n)$. If m and n are both even, then we have $m/2 = n/2$, so $m = n$. Similarly, if m and n are both odd, $-\frac{m+1}{2} = -\frac{n+1}{2}$ so $m = n$. We can not have m even and n odd, since then $f(m) \geq 0$ and $f(n) < 0$, so $f(m) \neq f(n)$. The case m odd and n even is similarly ruled out. Therefore $f(m) = f(n)$ implies $m = n$, so f is one-to-one.

To see that f is onto, let $z \in \mathbb{Z}$. If $z \geq 0$, then $2z \in \mathbb{J} \cup \{0\}$ and $f(2z) = z$. If $z < 0$ then $-2z - 1 \in \mathbb{J}$ and $f(-2z - 1) = z$. Therefore f is onto $\mathbb{Z}$, so f is a bijection from $\mathbb{J} \cup \{0\}$ to $\mathbb{Z}$.

So $\mathbb{J} \cup \{0\} \sim \mathbb{Z}$.

Exercise 1.4.

Lemma. *If $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is a bijection.*

Proof. We need to show that f^{-1} is one-to-one and onto.
Let $x, y \in B$ such that $f^{-1}(x) = f^{-1}(y)$. Then $x = f(f^{-1}(x)) = f(f^{-1}(y)) = y$. So f^{-1} is one-to-one.
Now let $a \in A$. Then $f(a) \in B$ and $f^{-1}(f(a)) = a$. Therefore f^{-1} maps B onto A, as desired. □

To show that equal cardinality is an equivalence relation we need to show that it is reflexive, symmetric, and transitive.
Reflexive: Given any set A the identity map $i : A \to A$ is a bijection, hence A has the same cardinality as A.
Symmetric: Given two sets A and B such that there exists a bijection $f : A \to B$, by the above lemma the inverse map $f^{-1} : B \to A$ is also a bijection. So equal cardinality is a symmetric relation.

Transitive: Given sets A, B, C such that there exist bijections $f : A \to B$ and $g : B \to C$, by exercise 2 the map $g \circ f : A \to C$ is also a bijection. Therefore equal cardinality is a transitive relation.

Exercise 1.5. Define $\mathcal{E}_n = \{m/n : m \in \mathbb{Z}\}$. Then $\mathbb{Q} = \cup_{n=1}^{\infty} \mathcal{E}_n$. Since each $\mathcal{E}_n$ has the cardinality of $\mathbb{Z}$, which is countable by exercise 3, we know that the $\mathcal{E}_n$ are countable. Then from Rudin 2.12 (proved in class) $\mathbb{Q}$, as a countable union of countable sets, is itself countable.

Exercise 1.6. Assume that $S \backslash K$ were at most countable. Then $S = K \cup S \backslash K$ would be the union of two at most countable sets and hence by the corollary to Rudin 2.12, S is at most countable, contradicting the fact that S is uncountable.

2 June 12, 2008

Exercise 2.1.

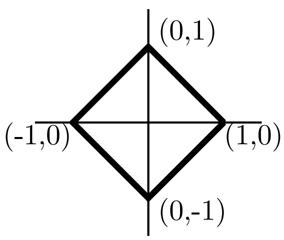
(a) (The following proof uses the same approach as Rudin's proof of Theorem 2.14).

Since $\mathbb{N}$ is infinite we know that the set of all subsets of $\mathbb{N}$ is infinite. Let E be any countable collection of subsets of $\mathbb{N}$. Label the elements of E as $s_1, s_2, s_3, \ldots$. Define a subset s of $\mathbb{N}$ by $s = \{n \in \mathbb{N} : n \notin s_n\}$. So for each $n \in \mathbb{N}$, $s \neq s_n$ since n is in either s or s_n but not both. Therefore, every countable subset of the set of all subsets of $\mathbb{N}$ is a proper subset, so the set of all subsets of $\mathbb{N}$ must be uncountable.

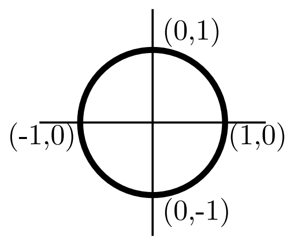
(b) Define a map $f : A \to \mathbb{Q}$ by $f(S) = .s_1s_2s_3\ldots$ where $s_i = 1$ if $i \in S$ and $s_i = 0$ if $i \notin S$. Since each $S \in A$ has a finite number of elements, these decimal expansions have a finite number of non-zero terms and hence are elements of $\mathbb{Q}$. If $f(S) = .s_1s_2s_3\ldots = .t_1t_2t_3\ldots = f(T)$ for $S, T \in A$, then we must have $s_i = t_i$ for all $i \in \mathbb{N}$, which by the definition of f implies that S and T have the same elements. Therefore f is one-to-one. So f is a bijection from A onto some subset of $\mathbb{Q}$. Since $\mathbb{Q}$ is countable, this implies that A is at most countable. Since $\mathbb{N}$ is infinite we know that there are an infinite number of finite subsets of $\mathbb{N}$. Therefore A is countable.

Exercise 2.2.

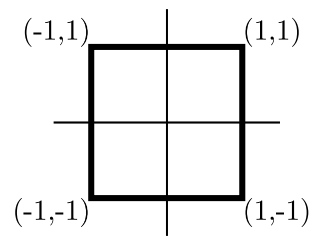
(a) $d((x,y),0) = |x| + |y|$



(b) $d((x,y),0) = \sqrt{x^2 + y^2}$



(c) $d((x,y),0) = \max\{|x|, |y|\}$



Exercise 2.3.

We will show that taxicab distance satisfies the three properties of a metric space. Let $a, b, c \in X^n$ with $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_n)$, and $c = (c_1, c_2, \ldots, c_n)$.

(a) If $a \neq b$, then there exists some $j \in \{1, \ldots, n\}$ such that $a_j \neq b_j$, so $d_X(a_j, b_j) > 0$. Since all of the other $d_X(a_i, b_i)$ are non-negative, $d(a,b) = \sum_{i=1}^n d_X(a_i, b_i) > 0$.

If $a = b$ then $a_i = b_i$ for all $i \in \{1, \ldots, n\}$, so $d_X(a_i, b_i) = 0$ for all i. Therefore $d(a,b) = \sum_{i=1}^n d_X(a_i, b_i) = 0$, as desired.

(b) The symmetry of d follows from the symmetry of d_X

$d(a,b) = \sum_{i=1}^n d_X(a_i, b_i) = \sum_{i=1}^n d_X(b_i, a_i) = d(b,a)$.

(c) The triangle inequality for d follows from the triangle inequality for d_X

$$\begin{aligned} d(a,b) &= \sum_{i=1}^{n} d_X(a_i,b_i) \\ &\leq \sum_{i=1}^{n}[d_X(a_i,c_i)+d_X(c_i,b_i)] \\ &= \sum_{i=1}^{n} d_X(a_i,c_i) + \sum_{i=1}^{n} d_X(c_i,b_i) \\ &= d(a,c)+d(c,b) \end{aligned}$$

Therefore (X^n,d) is a metric space.

Exercise 2.4.

Let $x,y,z \in S$.
If $x \neq y$ then $d(x,y) > 0$ so $\tilde{d}(x,y) = \frac{d(x,y)}{1+d(x,y)} > 0$.
If $x = y$ then $d(x,y) = 0$ so $\tilde{d}(x,y) = \frac{0}{1+0} = 0$
Since d is symmetric, $\tilde{d}(x,y) = d(x,y)/(1+d(x,y)) = d(y,x)/(1+d(y,x)) = \tilde{d}(y,x)$, so $\tilde{d}$ is symmetric.
Note that for $a,b \geq 0, a \leq b$ we have

$$a+ab \leq b+ab \quad \Rightarrow \quad a(1+b) \leq b(1+a) \quad \Rightarrow \quad a/(1+a) \leq b/(1+b).$$

Applying this result to $d(x,y) \leq d(x,z)+d(z,y)$ we have

$$\begin{aligned} \tilde{d}(x,y) = \frac{d(x,y)}{1+d(x,y)} &\leq \frac{d(x,z)+d(y,z)}{1+d(x,z)+d(y,z)} \\ &= \frac{d(x,z)}{1+d(x,z)+d(y,z)} + \frac{d(y,z)}{1+d(x,z)+d(y,z)} \\ &\leq \frac{d(x,z)}{1+d(x,z)} + \frac{d(y,z)}{1+d(y,z)} \\ &= \tilde{d}(x,z)+\tilde{d}(z,y). \end{aligned}$$

The second inequality holds since $d(x,z)$ and $d(y,z)$ are both non-negative. This proves the triangle inequality for $\tilde{d}$. Therefore $(S,\tilde{d})$ is a metric space.

Exercise 2.5.

Let $a,b,c \in R$ such that $0 < a \leq b \leq c$ and $a+b \geq c$. Define $d(x_i,x_i) = 0$ for $i = 1,2,3$. Let $d(x_1,x_2) = d(x_2,x_1) = a$, $d(x_2,x_3) = d(x_3,x_2) = b$, and $d(x_1,x_3) = d(x_3,x_1) = c$. Then $d(x_i,x_j) = 0$ when $i = j$ and $d(x_i,x_j) > 0$ when $i \neq j$. By our construction d is symmetric. To verify the triangle inequality, note that if i,j,k are distinct then $d(x_i,x_j) \leq c \leq a+b \leq d(x_i,x_k)+d(x_k,x_j)$. In the case $i = j$, $d(x_i,x_j) = 0 \leq d(x_i,x_k)+d(x_k,x_j)$. If $i = k$ then $d(x_i,x_j) = d(x_k,x_j) = d(x_i,x_k)+d(x_k,x_j)$, and the case $j = k$ is handled similarly. Therefore d satisfies the triangle inequality and is a metric for the space $\{x_1,x_2,x_3\}$.

Exercise 2.6.

I will show the desired equivalence by proving that $\frac{1}{\sqrt{n}} \cdot d(x,y) \leq d_\infty(x,y) \leq d(x,y)$ for all $x, y \in \mathbb{R}^n$. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Then

$$\begin{aligned}
\frac{1}{\sqrt{n}} \cdot d(x,y) &= \frac{1}{\sqrt{n}} \sqrt{\sum_{i=1}^{n} |y_i - x_i|^2} \\
&\leq \frac{1}{\sqrt{n}} \sqrt{n \cdot \max\{|y_i - x_i|^2\}_{i=1}^{n}} \\
&= \max\{|y_i - x_i|\}_{i=1}^{n} \\
&= d_\infty(x,y)
\end{aligned}$$

proving the first inequality. For the second inequality we have

$$\begin{aligned}
d_\infty(x,y) &= \max\{|y_i - x_i|\}_{i=1}^{n} \\
&= \sqrt{\max\{|y_i - x_i|^2\}_{i=1}^{n}} \\
&\leq \sqrt{\sum_{i=1}^{n} |y_i - x_i|^2} \\
&= d(x,y)
\end{aligned}$$

Therefore d and d_∞ are equivalent.

Exercise 2.7.

We begin by showing that d is a metric. Positivity follows directly from the definition of d. The symmetry of d follows from the symmetry of the relations $p = q$ and $p \neq q$. For the triangle inequality, let $p, q, r \in X$. If $p = q$, then since d is a non-negative function we have

$$d(p,q) = 0 \leq d(p,r) + d(r,q).$$

In the case where $p \neq q$, by transitivity we must have either $p \neq r$ or $q \neq r$ (or both), so

$$d(p,q) = 1 \leq d(p,r) + d(r,q).$$

Therefore d is a metric on X. Note that the neighborhoods $N_\epsilon(x)$ in this metric are either the point x (when $\epsilon \leq 1$) or all of X (when $\epsilon > 1$).

Let S be any subset of X and let p be any point of S. Then $N_{1/2}(p) = \{p\} \subseteq S$. Therefore p is an interior point of S. Since this holds for all $p \in S$, S is an open set.

Since any subset of X is open, the complement of any subset of X is open, so any subset of X is also closed. Alternatively, we can show that any subset of X is closed by noting that any subset of X has no limit points. This follows since any point $p \in X$ has a neighborhood $N_{1/2}(p) = \{p\}$ that contains no points of X except for p itself.

Exercise 2.8.

Let $S \subseteq \mathbb{R}^n$ be an open set and let $p = (p_1, p_2, \ldots, p_n) \in S$. Fix any $\epsilon > 0$ and consider the neighborhood $N_\epsilon(p)$. Since S is open there exists some $\delta > 0$ such that $N_\delta(p) \subseteq S$. Fix any $\alpha > 0$ such that α is less than both ϵ and δ. Then the point $q = (p_1 + \alpha, p_2, \ldots, p_n) \neq p$, $q \in N_\delta(p) \subseteq S$, and $q \in N_\epsilon(p)$. Since ϵ was arbitrary, this shows that every neighborhood of p contains an element of S not equal to p. Therefore p is a limit point of S, so S has no isolated points.

Note that this result depended on being in $\mathbb{R}^n$ in order to guarantee the existence of the point q. In other metric spaces, for example the previous exercise, open sets can have isolated points.

Exercise 2.9.

(a) Let $A = [0, 1)$.

Then A is not open since for every $\epsilon > 0$, the neighborhood $N_\epsilon(0) = (-\epsilon, \epsilon) \nsubseteq [0, 1)$.

A is not closed because $1 \notin A$, but 1 is a limit point of A (for every $\epsilon > 0$, $1 - \epsilon/2 \in A$ and $1 - \epsilon/2 \in N_\epsilon(1)$).

(b) $B = (1, 2) \cup \{3\}$. B is not open because 3 is not an interior point. B is not closed because 1 and 2 are limit points of B not contained in B.

(c) $C = \{1/n : n \in \mathbb{J}\}$. C is not open because 1 is not an interior point (in fact no points in C are interior points). C is not closed because 0 is a limit point of C not contained in C (for every $\epsilon > 0$, choose $n \in \mathbb{J}$ such that $1/n < \epsilon$. Then $1/n \in N_\epsilon(0)$).

Exercise 2.10.

By exercise 2.8, every open set in $\mathbb{R}^2$ has no isolated points. Therefore, every point of every open set must be a limit point.

The same is not true for closed sets. Consider the set of any single point $p \in \mathbb{R}^2$. This set has no limit points, so it contains all of its limit point and is therefore closed. However, the point p is not a limit point.

3 June 16, 2008

Exercise 3.1.

Let (X, d) be a metric space and let B be the closed neighborhood of radius $r > 0$ about a point $p \in X$. Consider any point $q \in X$ such that $d(p,q) > r$. Let $\delta = \frac{1}{2}(d(p,q) - r)$. Then by the triangle inequality, $B \cap N_\delta(q) = \emptyset$. Therefore q is not a limit point of B. So all the limit points of B must be a distance at most r away from p. Therefore all the limit points of B are in B, and B is closed.

Exercise 3.2.

Let x be a limit point of E'. Fix any $\epsilon > 0$. Then there exists a point $p \in E'$ such that $p \in N_{\epsilon/2}(x)$. Let $\delta = \min\{\epsilon/2, d(x,p)\}$. Since p is a limit point of E, there exists a point $q \in E$ such that $q \in N_\delta(p)$. By the triangle inequality $d(x,q) \geq d(x,p) - d(p,q) > 0$, so $x \neq q$. Also by the triangle inequality we have $d(x,q) \leq d(x,p) + d(p,q) < \epsilon$, so $q \in N_\epsilon(x)$. Since ϵ was arbitrary, this shows that x is a limit point of E, so $x \in E'$. Therefore E' is closed.

All the limit points of E need not be limit points of E'. For example, consider $E = \{1/n : n \in \mathbb{J}\}$ as a subset of $\mathbb{R}$ with the Euclidean metric. Then $E' = \{0\}$, so E' has no limit points. Therefore 0 is a limit point of E which is not a limit point of E'.

Exercise 3.3.

The arguments for the interval [-8,-4) and the points $\{-2, 0\}$ mostly follow from exercise 2.9, so here I concentrate on the arguments for $(\mathbb{Q} \cap (1, 2\sqrt{2}))$.

(a) Since the irrationals are dense in $\mathbb{R}$, for any point $p \in \mathbb{Q} \cap (1, 2\sqrt{2})$ any neighborhood of p will contain irrationals that are not in X and therefore p is not an interior point of X. So $\text{Int}(X) = (-8, -4)$.

(b) For any point $q \in ([1, 2\sqrt{2}] \backslash \mathbb{Q})$, since the rationals are dense in $\mathbb{R}$ any neighborhood of q will contain elements of X, so q can not be in the exterior of X. So $\text{Ext}(X) = (-\infty, 8) \cup (-4, -2) \cup (-2, 0) \cup (0, 1) \cup (2\sqrt{2}, \infty)$.

(c) As argued in the previous part, every point in $[1, 2\sqrt{2}]$ is a limit point of X. So $\bar{X} = [-8, -4] \cup \{-2, 0\} \cup [1, 2\sqrt{2}]$.

(d) $\partial X = \bar{X} \backslash \text{Int}(X)$. So in this case $\partial X = \{-8, -4, -2, 0\} \cup [1, 2\sqrt{2}]$.

(e) Since $\mathbb{Q}$ is dense in $\mathbb{R}$, every neighborhood of element of $(\mathbb{Q} \cap (1, 2\sqrt{2}))$ contains other elements of $(\mathbb{Q} \cap (1, 2\sqrt{2}))$. Therefore no point of $(\mathbb{Q} \cap (1, 2\sqrt{2}))$ is an isolated element of X. So the set of isolated points of X is $\{-2, 0\}$.

Exercise 3.4.

Since U is open, for any point $x \in U$, there exists a neighborhood of x completely contained in U, so $x \in \text{Int}(U)$. Therefore $x \notin \partial U$. So $U \cap \partial U = \emptyset$. Since $U \subseteq \bar{U}$, this gives $U \subseteq \bar{U} \backslash \partial U$.

Conversely, let $y \in \bar{U}$ and assume $y \notin U$. Then $y \notin \mathrm{Int}(U)$, hence $y \in \partial U = \bar{U} \backslash \mathrm{Int}(U)$. So $\bar{U} \backslash \partial U \subseteq U$. Therefore $U = \bar{U} \backslash \partial U$.

If U is not open, then the above result does not hold. If $x \in U$ is not an interior point of U, then $x \in \partial U$, so $x \notin \bar{U} \backslash \partial U$. For example, if $U = [0, 1]$ in $\mathbb{R}$ with the Euclidean metric, then $\bar{U} \backslash \partial U = (0, 1)$.

Exercise 3.5.

(a) Let $x \in D(A, \epsilon)$. Then $d(x, A) < \epsilon$. Choose $\delta > 0$ such that $d(x, A) + \delta < \epsilon$. Then from the definition of $d(x, A)$ there exists $a \in A$ such that $d(x, a) < d(x, A) + \delta$. Now let $\gamma = (\epsilon - \delta - d(x, A)) > 0$. Then by the triangle inequality, for any $y \in N_\gamma(x)$, $d(y, a) < \epsilon$, so $d(y, A) \leq d(y, a) < \epsilon$. So $y \in D(A, \epsilon)$. Therefore $N_\gamma(x) \subseteq D(A, \epsilon)$, so $D(A, \epsilon)$ is open.

(b) We will show that N_ϵ is closed by showing that its compliment is open. Let $y \in M \backslash N_\epsilon$. Then $d(y, A) > \epsilon$. So for every $a \in A$, $d(y, a) > \epsilon$. Let $\delta = \frac{1}{2}(d(y, a) - \epsilon) > 0$. Let $z \in N_\delta(y)$. Then by the triangle inequality, for all $a \in A$ we have $d(z, a) > \frac{1}{2}(3\epsilon - d(y, a))$. Therefore $d(z, A) \geq \frac{1}{2}(3\epsilon - d(y, a)) > \epsilon$. So $z \in M \backslash N_\epsilon$. Therefore $M \backslash N_\epsilon$ is open, so N_ϵ is closed.

We now show that A is closed $\Leftrightarrow A = \cap_\epsilon N_\epsilon$.

$\Rightarrow$ If $x \in A$, then $d(x, A) = 0$, so $x \in N_\epsilon$ for all $\epsilon > 0$. Therefore $A \subseteq \cap_\epsilon N_\epsilon$.

To show the converse, let $y \in \cap_\epsilon N_\epsilon$. Then $d(y, A) \leq \epsilon$ for all $\epsilon > 0$. Therefore $d(y, A) = 0$. So for every $\delta > 0$, there exists an $a \in A$ such that $d(y, a) < \delta$. Then every neighborhood $N_\delta(y)$ contains an element of A, so either $y \in A$, or y is a limit point of A. Since A is closed, in both cases we have $y \in A$, so $\cap_\epsilon N_\epsilon \subseteq A$.

$\Leftarrow$ Now assume $A = \cap_\epsilon N_\epsilon$. From above we know that the N_ϵ are all closed, so we have that A is an intersection of closed sets, hence A is closed.

Exercise 3.6.

(a) The set $\mathbb{Z} \subseteq \mathbb{R}$ is infinite but has no limit points.

(b) The set of limit points of the interval (0,1) is the interval [0,1]. So (0,1) is (properly) contained in its set of limit points.

(c) Let $A = \{m + \frac{1}{n+1} : m, n \in \mathbb{J}\}$. Then $\mathbb{J}$ is the set of limit points of A, but $A \cap \mathbb{J} = \emptyset$.

(d) Let A be a single point in $\mathbb{R}$. Then A has no limit points so $\bar{A} = A$. $\mathrm{Int}A = \emptyset$, so $\partial A = A$.

Exercise 3.7.

We will show that x_n is Cauchy and therefore converges in $\mathbb{R}^k$. Fix any $\epsilon > 0$. By repeated application of the relation $d(x_{n+1}, x_n) \leq r \cdot d(x_n, x_{n-1})$ we have $d(x_{n+m}, x_{n+m-1}) \leq r^m \cdot d(x_n, x_{n-1})$ for all $m \in \mathbb{J}$. Since $0 \leq r < 1$, the sequence r^m converges to zero. Therefore, the sequence $d(x_m, x_{m-1})$ will also converge to zero. So there exists $N \in \mathbb{J}$ sufficiently large such that $d(x_N, x_{N-1}) < \epsilon(1-r) > 0$. Then for $n > m > N$ we have

$$\begin{aligned} d(x_n, x_m) &= \sum_{i=1}^{n-m} d(x_{i+m}, x_{i+m-1}) \\ &\leq \sum_{i=1}^{\infty} d(x_{i+m}, x_{i+m-1}) \\ &\leq \sum_{i=1}^{\infty} r^i \cdot d(x_m, x_{m-1}) \\ &\leq \sum_{i=1}^{\infty} r^i \cdot d(x_N, x_{N-1}) \\ &< d(x_N, x_{N-1}) \cdot \frac{1}{1-r} \\ &< \epsilon \end{aligned}$$

Therefore x_m is Cauchy, as desired.

Exercise 3.8.

Let $\{s_n\}$ be a convergent sequence in $\mathbb{C}$ and let s be the limit of $\{s_n\}$. Fix $\epsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that for all $n > N$, $|s_n - s| \leq \epsilon$. By the triangle inequality, for all $n \in \mathbb{N}$ we have

$$\begin{aligned} |s_n - 0| \leq |s_n - s| + |s - 0| \quad &\Rightarrow \quad |s_n| - |s| \leq |s_n - s| \\ |s - 0| \leq |s_n - s| + |s_n - 0| \quad &\Rightarrow \quad |s| - |s_n| \leq |s_n - s| \end{aligned}$$

Combining these two inequalities, for $n > N$ we have

$$||s_n| - |s|| \leq |s_n - s| \leq \epsilon$$

Since ϵ was arbitrary, this implies $\{|s_n|\} \to |s|$, as desired.

The converse statement does not hold in general. For example, consider the sequence $s_n = (-1)^n$. In this case $\{|s_n|\}$ converges to 1 while $\{s_n\}$ diverges.

Exercise 3.9.

Let $y \notin A$. Then $d(x, y) > 0$. So let $\epsilon = d(x, y)/3 > 0$. Since $\{x_n\}$ converges to x, there exists some $N \in \mathbb{J}$ such that for $n > N$, $x_n \in N_\epsilon(x)$. Let $\delta = \min\{\epsilon, d(y, x_i)\}_{i=1}^N$. Then by the triangle inequality, $N_\delta(y) \cap N_\epsilon(x) = \emptyset$ and by our choice of δ, $x_i \notin N_\delta(y)$ for $i = \{1, 2, \ldots, N\}$. So $N_\delta(y) \cap A = \emptyset$. Therefore y is not a limit point of A, so A contains all of its limit points and is therefore closed.

Alternatively, Let y be a limit point of A such that $y \notin A$. We will inductively define a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges to y. Let $x_{n_1} = x_1$. Given $\{x_{n_i}\}_{i=1}^k$, let $\delta_k = \min\{1/(k+1), d(y, x), d(y, x_j)\}_{j=1}^{n_k} > 0$ (this is positive since $y \notin A$). Since y is a limit point of A, the neighborhood $N_{\delta_k}(y)$ must contain some element of A. By our choice of δ_k, any such element must be x_j for some $j > n_k$. So define $x_{n_{k+1}}$ to be any element in $N_{\delta_k}(y) \cap A$. Since $d(y, x_{n_i}) < 1/i$ for all $i \in \mathbb{J}$, the subsequence $\{x_{n_i}\}$ converges to y. But $\{x_n\}$ converges to x, and any subsequence of a convergent sequence converges to the same limit. Therefore $y = x \in A$. So A must contain all of its limit points and is therefore closed.

4 June 18, 2008

Exercise 4.1.

Let $L > 1/2$. Then for even $n > 2$, $x_n < 1/4$, so $x_n \notin N_{1/4}(L)$. So there does not exist $N \in \mathbb{N}$ such that for all $n > N$, $x_n \in N_{1/4}(L)$. Therefore $\{x_n\}$ does not converge to L.

Now let $L \leq 1/2$. Then for odd $n > 2$, $x_n > 4/5$, so $x_n \notin N_{1/4}(L)$. So as argued above, $\{x_n\}$ does not converge to L.

Therefore $\{x_n\}$ does not converge to any real number, hence the sequence diverges.

Exercise 4.2.

(a) Since P, Q are non-empty and bounded above, $\sup Q$ and $\sup P$ exist in $\mathbb{R}$. If $p \in P$, then $p \in Q$, so $p \leq \sup Q$ since $\sup Q$ is an upper bound for Q. Therefore $\sup Q$ is also an upper bound for P. Since $\sup P$ is the least upper bound of P, this implies that $\sup P \leq \sup Q$.

(b) Let $A = \limsup_{n\to\infty} a_n$ and $B = \limsup_{n\to\infty} b_n$. If $\{A, B\} = \{\pm\infty\}$, then the right hand side of the desired inequality is undefined, so we can not prove the result in this case. If at least one of A, B is infinity and neither is $-\infty$, then the right hand side of the desired inequality will be infinite and hence the inequality holds.

So we are left with only the case where both A and B are not positive infinity. Fix any number $L > A + B$. Let $\delta = \frac{1}{2} \cdot (L - (A + B)) > 0$. Then by theorem 3.17, there exist $N_1, N_2 \in N$ such that for $n > N_1$, $a_n < A + \delta/2$ and for $n > N_2$, $b_n < B + \delta/2$. So for $n > N := \max\{N_1, N_2\}$ we have $a_n + b_n < A + B + \delta = L - \delta$. Therefore, no subsequence of $\{a_n + b_n\}$ can converge to L. So $\limsup_{n\to\infty}(a_n + b_n) \leq A + B = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.

Exercise 4.3.

(a) Fix any $\epsilon > 0$. Then since $\{a_n\}$ converges to a, there exists a $N_1 \in \mathbb{J}$ such that for $n > N_1$, $|a - a_n| < \epsilon/2$. Let $A_{N_1} = \sum_{i=1}^{N_1}(a_i - a)$. Since A_{N_1} is finite, there exists $N \in \mathbb{J}$, $N \geq N_1$, such that for $n > N$, $|A_{N_1}|/n < \epsilon/2$. Then for $n > N$ we have

$$
\begin{aligned}
|x_n - a| &= \left|\frac{a_1 + a_2 + \cdots + a_n}{n} - \frac{n \cdot a}{n}\right| \\
&= \left|\frac{(a_1 - a) + (a_2 - a) + \cdots + (a_n - a)}{n}\right| \\
&= \left|\frac{(a_1 - a) + (a_2 - a) + \cdots + (a_N - a)}{n} + \frac{(a_{N+1} - a)}{n} + \frac{(a_{N+2} - a)}{n} + \cdots + \frac{(a_n - a)}{n}\right| \\
&\leq \left|\frac{(a_1 - a) + (a_2 - a) + \cdots + (a_N - a)}{n}\right| + \left|\frac{(a_{N+1} - a)}{n}\right| + \left|\frac{(a_{N+2} - a)}{n}\right| + \cdots + \left|\frac{(a_n - a)}{n}\right| \\
&< \epsilon/2 + (n - N)\frac{\epsilon/2}{n} \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon
\end{aligned}
$$

Therefore $\{x_n\}$ converges to a as desired.

(b) Let $a_n = (-1)^n$ Then $a_1 + a_2 + \cdots + a_n$ is -1 when n is odd and zero when n is even. Hence x_n is $-1/n$ when n is odd and zero when n is even. Since $-1/n$ converges to zero, the entire sequence x_n converges to zero.

Exercise 4.4.

(a) Let $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \alpha < 1$ and fix any number $b \in (\alpha, 1)$. Then there exists an $N \in \mathbb{N}$ such that for all $n > N$, $\frac{x_{n+1}}{x_n} < b$. So for $n > N$ we have $x_{n+1} < b \cdot x_n$. Applying this repeatedly we have for any $n > N$

$$x_n < b \cdot x_{n-1} < b^2 \cdot x_{n-2} < \cdots < b^{n-N} \cdot x_N = b^n \cdot b^{-N} \cdot x_N.$$

Since $b \in (\alpha, 1)$, the sequence b^n converges to zero. Since $b^{-N} \cdot x_N$ is a finite constant, the sequence $b^n \cdot b^{-N} \cdot x_N$ will converge to $0 \cdot (b^{-N} \cdot x_N) = 0$. Therefore $\{x_n\}$ converges to zero, as desired.

(b) Let $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \alpha > 1$ and fix any number $b \in (1, \alpha)$. Then there exists an $N \in \mathbb{N}$ such that for all $n > N$, $\frac{x_{n+1}}{x_n} > b$. So for $n > N$ we have $x_{n+1} > b \cdot x_n$. Applying this repeatedly we have for any $n > N$

$$x_n > b \cdot x_{n-1} > b^2 \cdot x_{n-2} > \cdots > b^{n-N} \cdot x_N = b^n \cdot b^{-N} \cdot x_N.$$

Since $b > 1$, the sequence b^n is unbounded and positive. Since $b^{-N} \cdot x_N$ is a finite positive constant, the sequence $b^n \cdot b^{-N} \cdot x_N$ is also unbounded and positive. Therefore $\{x_n\}$ is unbounded and hence is not convergent.

(c) Let $y_n = 1$. Then $\{y_n\}$ converges to 1 and $\lim_{n\to\infty} \frac{y_{n+1}}{y_n} = 1$. Let $z_n = n$. Then $\{z_m\}$ is unbounded and hence divergent, but $\lim_{n\to\infty} \frac{z_{n+1}}{z_n} = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1$.

Exercise 4.5.

Let $\{x_n\}$ be a Cauchy sequence and let $\{x_{n_i}\}$ be a subsequence converging to a limit x. Let $\{x_{m_i}\}$ be any subsequence of $\{x_n\}$. Fix any $\epsilon > 0$. Since $\{x_n\}$ is Cauchy there exists $N \in \mathbb{N}$ such that for $n, m > N$, $d(x_n, x_m) < \epsilon/2$. Since $\{x_{n_i}\}$ converges to x there exists $N' \in \mathbb{N}$ such that for $i > N'$, $d(x_{n_i}, x) < \epsilon/2$. Fix any element x_{n_k} of the convergent subsequence such that $n_k > N$ and $k > N'$. Then for all i such that $m_i > N$ we have

$$d(x_{m_i}, x) \leq d(x_{m_i}, x_{n_k}) + d(x_{n_k}, x) \leq \epsilon/2 + \epsilon/2 = \epsilon$$

Therefore $\{x_{m_i}\}$ also converges to x.

Note that in the above proof we did not assume that the second subsequence $\{x_{m_i}\}$ converges. We have therefore proved the stronger result that if any subsequence of a Cauchy sequence converges to a point x, then all subsequences converge to x. In particular, the original sequence also converges to x.

Exercise 4.6.

$\Rightarrow$ If $\{x_n\}$ converges to x then we know that every subsequence of $\{x_n\}$ converges to x. Since a subsequence of a subsequence of $\{x_n\}$ is still a subsequence of $\{x_n\}$, we have that every subsequence of every subsequence of $\{x_n\}$ converges to x.

$\Leftarrow$ We prove the reverse direction by showing the contrapositive. Assume that $\{x_n\}$ does not converge to $\{x\}$. Then there exists $\epsilon > 0$, such that for all $N \in \mathbb{N}$ there exists $n > N$ such that $x_n \notin N_\epsilon(x)$. We inductively define a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that has no further subsequence converges to x. Let $x_{n_1} = x_1$. Given x_{n_k}, we know that there exists $m > n_k$ such that $x_m \notin N_\epsilon(x)$. Let $x_{n_{k+1}} = x_m$. Then since $d(x_{n_i}, x) \geq \epsilon$ for all $i \in \mathbb{N}$, no subsequence of $\{x_{n_i}\}$ can converge to x.

Exercise 4.7.

Let $\{x_n\} \subseteq K$ be a Cauchy sequence. Then since X is complete we know that $\{x_n\}$ converges to some point $x \in X$. If there exist $k \in N$ such that $x_k = x$, then since $x_k \in K$, $x \in K$. Otherwise, since $\{x_n\} \subseteq K$ converges to x, every neighborhood of x contains a point $x_n \neq x$, $x_n \in K$, so x is a limit point of K. Then since K is closed, $x \in K$. Therefore in both cases we have that $\{x_n\}$ converges to a point in K. So every Cauchy sequence in K converges in K, so K is complete.

Exercise 4.8.

We first note that every Cauchy sequence is bounded. Let $\{x_n\}$ be a Cauchy sequence in X. Then there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $d(x_n, x_m) \leq 1$. Let $M = \max\{1, d(x_N, x_i)\}_{i=1}^{N-1}$. Then $d(x_N, x_i) \leq M$ for all $i \in \mathbb{N}$. Therefore $\{x_n\}$ is bounded.

Since any subsequence of a bounded sequence is also bounded, every subsequence of $\{x_n\}$ is bounded. Hence every subsequence of $\{x_n\}$ has at least one subsequential limit point. Since a subsequence of a subsequence of $\{x_n\}$ is still a subsequence of $\{x_n\}$, all of these subsequential limit points must be the same by exercise 4.6. Call this common limit point x. Then by exercise 4.7, since every subsequence of $\{x_n\}$ has a further subsequence converging to x we know that $\{x_n\}$ also converges to x. Therefore every Cauchy sequence in X converges, so X is complete.

Alternatively, once we know that any Cauchy sequence $\{x_n\}$ is bounded, we know that $\{x_n\}$ has some subsequential limit point. Then by the remark following the proof of exercise 4.6 we have that $\{x_n\}$ converges.

Exercise 4.9.

Let $\epsilon_n = 1 - \frac{1}{n}$.

We first show that $\{N_{\epsilon_n}((0,0))\}_{n=2}^{\infty}$ covers $\{(x,y) : x^2+y^2 < 1\}$. Fix any point $(a,b) \in \{(x,y) : x^2+y^2 < 1\}$ and let $r = d((a,b),(0,0)) = \sqrt{a^2+b^2} < 1$. Since $r < 1$, there exists $N \in \mathbb{N}$ such that $\epsilon_N = 1 - 1/N > r$. Then $(a,b) \in N_{\epsilon_N}((0,0))$.

Second we show that any finite subset of $\{N_{\epsilon_n}((0,0))\}_{n=2}^{\infty}$ does not cover $\{(x,y) : x^2+y^2 < 1\}$. Any finite subset of $\{N_{\epsilon_n}((0,0))\}_{n=2}^{\infty}$ will have a greatest $N \in \mathbb{N}$ such that $N_{\epsilon_N}((0,0))$ is in the subset. For $n > m$, $\epsilon_n > \epsilon_m$, so $N_{\epsilon_n}((0,0)) \supseteq N_{\epsilon_m}((0,0))$. Therefore the entire finite subset will be contained in $N_{\epsilon_N}((0,0))$. So the point $(\epsilon_N, 0)$ is in $\{(x,y) : x^2+y^2 < 1\}$ but is not in the finite subset.

So $\{N_{\epsilon_n}((0,0))\}_{n=2}^{\infty}$ is an open cover of $\{(x,y) : x^2 + y^2 < 1\}$ with no finite subcover. Therefore $\{(x,y) : x^2 + y^2 < 1\}$ is not compact.

Exercise 4.10.

Let $E \subseteq M$ be a subset containing a infinite number of points. Form an open cover of E by taking $\{N_{1/2}(x)\}_{x\in E} = \{x\}_{x\in E}$. Since E is infinite this is an infinite collection of sets and removing any set $N_{1/2}(x)$ from the collection results in the point $x \in E$ no longer being an element of any set in the collection. Therefore no finite sub-cover exists, so E is not compact.

Any finite set of points in any metric space is compact, so this result gives a complete characterization of the compact sets in a metric space with the discrete metric.

Exercise 4.11.

Here we sketch the solutions to this problem using the results already proved about the Cantor set in Rudin.

(a) Since $\mathbf{F}$ is an intersection of closed sets it is closed.

(b) In section 2.44, Rudin proves that $\mathbf{F}$ contains no intervals. Since open neighborhoods in $\mathbb{R}$ are open intervals, no open neighborhood of any point of $\mathbf{F}$ is contained in $\mathbf{F}$, hence $\mathbf{F}$ has no interior points.

(c) As argued for part (b), any non-empty open set in $\mathbb{R}$ contains an open interval and hence can not be contained in $\mathbf{F}$.

(d) Since $\mathbf{F}$ has no interior points, $\partial\mathbf{F} = \bar{\mathbf{F}}\backslash\text{Int}\mathbf{F} = \bar{\mathbf{F}} \supseteq \mathbf{F}$. So all points of $\mathbf{F}$ are in the boundary of $\mathbf{F}$.

(e) Since $\mathbf{F}$ contains no closed intervals it can not be written as a union of closed intervals.

(f) Using the notation of Rudin section 2.44, we have $\mathbf{F} = \cap_{n=1}^{\infty} E_n$ where each E_n is a union of a finite number of disjoint closed intervals each separated by an open interval. So $\mathbb{R} - E_n$ is a finite union of open intervals. Then by DeMorgan's law $\mathbb{R} - \mathbf{F} = \mathbb{R} - \cap_{n=1}^{\infty} E_n = \cup_{n=1}^{\infty}(\mathbb{R} - E_n)$. Since a countable union of finite sets is countable, this expresses the complement of $\mathbf{F}$ as a countable union of open intervals.

   For an alternative approach, since $\mathbf{F}$ is closed, the complement of $\mathbf{F}$ is open. The result then follows from the fact that any open set in $\mathbb{R}$ is a union of a countable collection of open intervals. Let $U \subset \mathbb{R}$ be an open set. Let $x \in U$. Then there exists $\epsilon > 0$ such that $N_\epsilon(x) \subseteq U$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ there exists $y \in (\mathbb{Q} \cap N_{\epsilon/2}(x))$. So $y \in U \cap \mathbb{Q}$ and $x \in N_{\epsilon/2}(y)$.

   For each point $y \in U \cap \mathbb{Q}$, let ϵ_y be the supremum of the radii of the neighborhoods centered at y given by the previous construction. Then $U = \cup_{y\in U\cap\mathbb{Q}} N_{\epsilon_y}(y)$, giving U as the desired countable union of open intervals.

5 June 23, 2008

Exercise 5.1.

Let $S \subseteq \mathbb{R}^n$ be an open and closed nonempty set and let S^c be non-empty. So we can find points $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$ such that $\mathbf{p} \in S$ and $\mathbf{q} \in S^c$. Recall that the line segment between $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{R}^n$ is given by the set of convex combinations $t\mathbf{p} + (1-t)\mathbf{q}$ where $t \in [0,1]$. So let A be the set of real numbers given by

$$A := \{t \in [0,1] : t\mathbf{p} + (1-t)\mathbf{q} \in S\}$$

A in non-empty since $0 \in A$ and A has an upper bound of 1, so by the least upper bound property of the real numbers $\sup A$ exists. Let $x := \sup A$ and define $\mathbf{r} = x\mathbf{p} + (1-x)\mathbf{q}$.

First consider the case $\mathbf{r} \in S$. By assumption $\mathbf{q} \in S^c$, so $x \in [0,1)$. Fix any $\epsilon > 0$ and then choose $s \in (x,1)$ such that $|s - x| < \frac{\epsilon}{d(\mathbf{p},\mathbf{q})}$. Define $\mathbf{u} = s\mathbf{p} + (1-s)\mathbf{q}$. Then

$$d(\mathbf{u}, \mathbf{r}) = |\mathbf{u} - \mathbf{r}| = |s\mathbf{p} + (1-s)\mathbf{q} - (x\mathbf{p} + (1-x)\mathbf{q})| = |(s-x)\mathbf{p} - (s-x)\mathbf{q}| = |(s-x)||\mathbf{p} - \mathbf{q}| < \epsilon.$$

Therefore $\mathbf{u} \in N_\epsilon(x)$. However, since $s > x$, $\mathbf{u} \notin S$. Since ϵ was arbitrary we have that no neighborhood of $\mathbf{r}$ is contained in S, so S is not open.

Second consider the case $\mathbf{r} \in S^c$. Fix any $\epsilon > 0$. Since $x = \sup A$ and $x \notin A$, there exists $t \in (x - \frac{\epsilon}{d(\mathbf{p},\mathbf{q})}, x)$ such that $t \in A$. Let $\mathbf{v} = t\mathbf{p} + (1-t)\mathbf{q} \in S$. Then repeating the calculation from the previous case we have $\mathbf{v} \in N_\epsilon(x)$. Therefore every neighborhood of x contains a point of S other than x itself. So x is a limit point of S. Since $x \notin S$, this implies that S is not closed.

Exercise 5.2.

In $\mathbb{R}^k$, define the k-cell centered at $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ with side length r to be

$$I_r(\mathbf{x}) = \{\mathbf{y} = (y_1, y_2, \ldots, y_k) \in \mathbb{R}^k : x_i - r/2 \le y_i \le x_i + r/2. \ \ i = 1, 2, \ldots, k\}.$$

I claim that for any $\epsilon > 0$, $N_\epsilon(\mathbf{x}) \supset I_{\epsilon/k}(\mathbf{x})$. To see this, let $\mathbf{y} \in I_{\epsilon/k}(\mathbf{x})$. Then for $i = 1, 2, \ldots, k$, $|y_i - x_i| \le \epsilon/2k$. Therefore

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{k} (x_i - y_i)^2} \le \sqrt{\sum_{i=1}^{k} \left(\frac{\epsilon}{2k}\right)^2} = \sqrt{\frac{\epsilon^2}{4k}} = \frac{\epsilon}{2\sqrt{k}} \le \epsilon,$$

with the last inequality following since $k \ge 1$. So $\mathbf{y} \in N_\epsilon(\mathbf{x})$, as desired.

Also note that since $d(\mathbf{x}, \mathbf{y}) \ge \max\{|x_i - y_i|\}_{i=1}^k$, for any $\epsilon > 0$ we have $N_\epsilon(\mathbf{x}) \subset I_{2\epsilon}(\mathbf{x})$.

Now let S be any bounded set in $\mathbb{R}^k$. So there exists an $M > 0$ such that $S \subseteq N_M(0)$. Fix any $\epsilon > 0$. Let $E' = \{n\epsilon/k : n \in \mathbb{Z}, \frac{(|n|-1)\epsilon}{k} \le M\}$. Then let $E = \{(x_1, x_2, \ldots, x_k) \in \mathbb{R}^k : x_i \in E'\}$. E is a rectangular lattice of points in $\mathbb{R}^k$ with a spacing of ϵ/k units between points, restricted to a k-cell of side length slightly larger than $2M$ centered at the origin. So E is a finite set. Then consider

$$\bigcup_{\mathbf{x} \in E} I_{\epsilon/k}(\mathbf{x}) \subset \bigcup_{\mathbf{x} \in E} N_\epsilon(\mathbf{x}). \tag{1}$$

The union of k-cells in (1) completely covers the k-cell of side length $2M$ centered at the origin and hence also covers $S \subseteq N_M(\mathbf{0}) \subset I_{2M}(\mathbf{0})$. Therefore the union of ϵ-neighborhoods in (1) also covers S. Since E is finite and k is finite, the union of ϵ-neighborhoods in (1) is finite. Therefore we have the desired finite covering by ϵ-neighborhoods.

Exercise 5.3.

Let (X, d) be a metric space. Let $\mathcal{E} \subseteq X$ be a set such that every infinite subset of $\mathcal{E}$ has a limit point in $\mathcal{E}$. We will inductively define a countable collection of finite sets which are dense in $\mathcal{E}$. Let $\epsilon_n := 1/n$. Fix any $k \in \mathbb{N}$. Fix any point $x_{k,1} \in \mathcal{E}$. Having chosen $x_{k,1}, x_{k,2}, \ldots, x_{k,j} \in \mathcal{E}$, choose $x_{k,j+1} \in \mathcal{E}$, if possible, such that $d(x_{k,j+1}, x_{k,i}) \geq \epsilon_k$ for $i = 1, 2, \ldots, j$. Note that by the triangle inequality, if we have two distinct points $x_{k,l}, x_{k,m} \in N_{\frac{1}{2}\epsilon_k}(x)$ for any point $x \in \mathcal{E}$ then $d(x_{k,l}, x_{k,m}) < \epsilon_k$, contradicting the construction of the $x_{k,i}$. Therefore, for each $x \in \mathcal{E}$ we have a neighborhood of x that contains a finite number of the points in $\{x_{k,i}\}$, so x is not a limit point of $\{x_{k,i}\}$. Since this holds for all $x \in \mathcal{E}$, $\{x_{k,i}\}$ has no limit points in $\mathcal{E}$. Therefore, by our assumption about the set $\mathcal{E}$, $\{x_{k,i}\}$ must have a finite number of elements. Let N_k be the size of the set $\{x_{k,i}\}$. Then consider the set:

$$D := \bigcup_{n \in \mathbb{N}} \{x_{n,i}\}_{i=1}^{N_n}.$$

I claim that D is a countable dense subset of $\mathcal{E}$. As a countable union of finite sets D is at most countable. Fix any point $x \in \mathcal{E}$ and any $\epsilon > 0$. Then fix $k \in N$ such that $\epsilon_k = 1/k < \epsilon$. Then there exists a point $x_{k,i} \in D$ such that $d(x_{k,i}, x) < \epsilon_k < \epsilon$ or else x would have been added to the set $\{x_{k,x}\}_{i=1}^{N_k}$. Since this holds for any $\epsilon > 0$, x is a limit point of D. Since this holds for all $x \in E$, the set D is dense in E.

A subset $\mathcal{E}$ of a metric space is *separable* if $\mathcal{E}$ contains a countable dense subset. So if every infinite subset of a set $\mathcal{E}$ has a limit point in $\mathcal{E}$ then $\mathcal{E}$ is separable. The converse of this statement is not true (consider $\mathcal{E} = \mathbb{R}$). Note that the idea of not being connected is NOT the same as being separable.

We will use the following lemma in the connected set problems.

Lemma. *Let X be a metric space and let $Z \subseteq Y \subseteq X$. If $A \subseteq Y$ is an open set the metric topology on Y then $A \cap Z$ is an open set in the metric topology on Z.*

Proof. Let $z \in A \cap Z$. Then $z \in A$ so there exists an $\epsilon > 0$ such that $N_\epsilon(z) \cap Y \subseteq A$. Then $N_\epsilon(z) \cap Z = N_\epsilon(z) \cap Y \cap Z \subseteq A \cap Z$. Therefore $A \cap Z$ is open in the metric topology on Z. □

Exercise 5.4.

Let $A = (-\infty, \sqrt{2}) \cap \mathbb{Q}$ and $B = (\sqrt{2}, \infty) \cap \mathbb{Q}$. Then $A \cup B = \mathbb{Q} \backslash \{\sqrt{2}\} = \mathbb{Q}$, and both A and B are non-empty. By the above lemma, since $(-\infty, \sqrt{2})$ and $(\sqrt{2}, \infty)$ are open in $\mathbb{R}$, both A and B are open sets in the metric topology on $\mathbb{Q}$. Therefore $\mathbb{Q}$ is not a connected subset of $\mathbb{R}$.

Exercise 5.5.

Proof by contrapositive. Let $\mathcal{F}$ be a family of connected subsets of a metric space X. Assume that $\cup_{F\in\mathcal{F}}F$ is not connected. Then there exist non-empty disjoint sets A, B such that $\cup_{F\in\mathcal{F}}F = A \cup B$ with A and B both open sets in the metric topology on $\cup_{F\in\mathcal{F}}F$.

Fix any set $F \in \mathcal{F}$. Then by the lemma, $A \cap F$ and $B \cap F$ are both open in the metric topology on F. Since A and B are disjoint, $A \cap F$ and $B \cap F$ are also disjoint. Since $\cup_{F\in\mathcal{F}}F = A \cup B$ we have $F = (A\cap F) \cup (B \cap F)$. Then since F is connected, either $A \cap F$ for $B \cap F$ is empty. So every $F \in \mathcal{F}$ is contained in either A or B. Since A and B are non-empty, we therefore have sets $F_1, F_2 \in \mathcal{F}$ such that $F_1 \subseteq A$ and $F_2 \subseteq B$. Since A and B are disjoint, F_1 and F_2 are disjoint. So we have two members of $\mathcal{F}$ without a common point, completing the contrapositive argument.

Exercise 5.6.

Assume that $\bar{S}$ is not connected. Then there exist non-empty disjoints sets A, B such that $A \cup B = \bar{S}$ where A and B are both open in the metric topology on $\bar{S}$. Then by the lemma, since $S \subseteq \bar{S}$, both $A \cap S$ and $B \cap S$ are open in the metric topology on S. Since A and B are disjoint, $A \cap S$ and $B \cap S$ are also disjoint, and since $\bar{S} = A \cup B$ we know that $S = (A \cap S) \cup (B \cap S)$.

We next show that $A \cap S$ is non-empty (an identical argument also applies to B). Since A is non-empty there exists a point $a \in A$. If $a \in S$ we are done, so assume $a \notin S$. Since $a \in \bar{S}$, this implies that $a \in \bar{S} \backslash S = S'$, the set of limit points of S. So every neighborhood of a contains a point of S. But since a is open in the metric topology on $\bar{S}$, there exists some $\epsilon > 0$ such that $N_\epsilon(a) \cap \bar{S} \subseteq A$. So $N_\epsilon(a) \cap S \subseteq A$, therefore A contains a point of S, so $A \cap S$ is non-empty.

Therefore $A \cap S$ and $B \cap S$ form a separation of S, so S is not connected, completing the contrapositive argument.

Exercise 5.7.

The proof of this problem is almost identical to the proof for the previous exercise with the set A replacing S and the set B replacing $\bar{S}$. The only change occurs in proving that any point in $B \backslash A$ is a limit point of A. Since $B \subseteq \bar{A}$, for any point $b \in B \backslash A$ we have $b \in \bar{A} \backslash A = A'$.

6 June 25, 2008

Exercise 6.1.

Fix any $\epsilon > 0$. Let $\epsilon' = \min\{L, \epsilon \cdot \sqrt{L}\}$ Then there exists $\delta > 0$ such that for all $x \in (a-\delta, a+\delta)\backslash\{a\}$, $|f(x) - L| < \epsilon'$. Then for $x \in (a-\delta, a+\delta)\backslash\{a\}$ we have $f(x) > 0$, so $\sqrt{f(x)} + \sqrt{L} > \sqrt{L}$. Therefore, for $x \in (a-\delta, a+\delta)\backslash\{a\}$ we have

$$\left|\sqrt{f(x)} - \sqrt{L}\right| = \left|\left(\sqrt{f(x)} - \sqrt{L}\right) \cdot \frac{\sqrt{f(x)} + \sqrt{L}}{\sqrt{f(x)} + \sqrt{L}}\right| = \frac{|f(x) - L|}{\left|\sqrt{f(x)} + \sqrt{L}\right|} < \frac{\epsilon \cdot \sqrt{L}}{\sqrt{L}} = \epsilon$$

Therefore $\lim_{x\to a} \sqrt{f(x)} = \sqrt{L}$.

Exercise 6.2.

(a) Assume that $\lim_{x\to p} f(x) = L$. Let M be any point in Y such that $L \neq M$. Let $\epsilon = \frac{1}{2} \cdot d(L, M) > 0$. Then there exists $\delta > 0$ such that for all $x \in E \cap N_{\delta_1}(p)\backslash\{p\}$, $f(x) \in N_\epsilon(L)$. By the triangle inequality, $N_\epsilon(L) \cap N_\epsilon(M) = \emptyset$. So for all $x \in E \cap N_\delta(p)\backslash\{p\}$, $f(x) \notin N_\epsilon(M)$. Fix any $\delta' > 0$. Let $\gamma = \min\{\delta, \delta'\}$. Since p is a limit point of E, there exists $y \in N_\gamma(p)\backslash\{p\} \subseteq N_\delta(p)\backslash\{p\}$. Then $y \in E \cap N_{\delta'}(p)\backslash\{p\}$ but $f(y) \notin N_\epsilon(M)$. Therefore $\lim_{x\to p} f(x) \neq M$. Therefore f has a unique limit at p.

(b) Assume that $\lim_{x\to p} f(x) = L$. Let M be any point in Y such that $L \neq M$. Let $\epsilon = \frac{1}{2} \cdot d(L, M) > 0$. Let $\{p_n\}$ be any sequence in E such that $p_n \neq p$ and $\lim_{n\to\infty} p_n = p$ (since p is a limit point of E, such a sequence exists). Then $f(p_n)$ converges to L, so there exists $N \in \mathbb{N}$ such that for all $n > N$, $f(p_n) \in N_\epsilon(L)$. By the triangle inequality, $N_\epsilon(L) \cap N_\epsilon(M) = \emptyset$. So for all $n > N$, $f(p_n) \notin N_\epsilon(M)$. Therefore $\{f(p_m)\}$ does not converge to M, so $\lim_{x\to p} f(x) \neq M$. Therefore f has a unique limit at p.

Exercise 6.3.

(a) Let $\{p_n\}$ be any sequence in E such that $p_n \neq p$ and $\lim_{n\to\infty} p_n = p$. Then $\lim_{n\to\infty} f(p_n) = A$. Additionally, $\lim_{n\to\infty} g(p_n) = B$. Then by Rudin Theorem 3.3, $\lim_{n\to\infty} f(p_n) \cdot g(p_n) = AB$. Therefore $\lim_{x\to p}(f \cdot g)(x) = AB$.

(b) Fix any $\epsilon > 0$. Let $\epsilon_1 = \frac{\epsilon}{|B|^3}$. Then there exists $\delta_1 > 0$ such that for all $x \in E \cap N_{\delta_1}(p)\backslash\{p\}$, $f(x) \in N_{\epsilon_1}(A)$. Let $A' = \max\{|A|, 1\}$. Define $\epsilon_2 = \min\{\frac{\epsilon}{|B|^2 \cdot A'}, \frac{|B|}{2}\}$. Then there exits $\delta_2 > 0$ such that for all $x \in E \cap N_{\delta_2}(p)\backslash\{p\}$, $f(x) \in N_{\epsilon_2}(B)$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then for $x \in E \cap N_\delta(p)\backslash\{p\}$ we have

$$\begin{aligned}\left|\frac{f(x)}{g(x)} - \frac{A}{B}\right| = \left|\frac{f(x) \cdot B - g(x) \cdot A}{g(x) \cdot B}\right| = \left|\frac{A(B - g(x)) + B(f(x) - A)}{g(x) \cdot B}\right| &\leq \frac{|A| \cdot |B - g(x)| + |B| \cdot |f(x) - A|}{|g(x)| \cdot |B|} \\ &< \frac{|A| \cdot \frac{\epsilon}{|B|^2 \cdot A'} + |B| \cdot \frac{\epsilon}{|B|^3}}{\frac{|B|}{2} \cdot |B|} \\ &\leq \frac{2\epsilon/|B|^2}{|B|^2/2} = \epsilon\end{aligned}$$

Therefore $\lim_{x\to p} \frac{f(x)}{g(x)} = \frac{A}{B}$.

In the preceding proof, A' is used to deal with the possibility that $A = 0$. Choosing $\epsilon_2 < |B|/2$ bounds $|g(x)|$ below by $|B|/2 > 0$.

Exercise 6.4.

Assume $L > M$. Let $\epsilon = \frac{1}{2}\cdot(L-M)$. Let $\{x_n\}$ be any sequence in A such that $x_n \neq p_0$ and $\lim_{n\to\infty} x_n = p$. Then $f(x_n)$ converges to L, so there exists N_1 such that for all $n > N_1$, $f(x_n) \in N_\epsilon(L)$. Similarly, $g(x_n)$ converges to M, so there exists N_2 such that for all $n > N_2$, $g(x_n) \in N_\epsilon(M)$. Let $N = \max\{N_1, N_2\}$. Then for $n > N$ we have

$$f(x_n) > L - \epsilon \geq M + \epsilon > g(x_n)$$

So we have found points $x_n \in A$, $n > N$, such that $f(x_n) > g(x_n)$, completing the proof of the contrapositive.

Exercise 6.5.

(a) Fix any point $(x, y) \in \mathbb{R}^2$ and any $\epsilon > 0$. For any point $(x_1, y_1) \in N_\epsilon((x, y))$ we have

$$\epsilon > \sqrt{(x-x_1)^2 + (y-y_1)^2} \geq \sqrt{(x-x_1)^2} = |x - x_1| = |f(x, y) - f(x_1, y_1)|$$

Therefore, f is continuous at (x, y). Since (x, y) was arbitrary, f is continuous on $\mathbb{R}^2$.

(b) I claim that $A = f^{-1}(U)$.

Let $(x, y) \in \mathbb{R}^2$. Then $(x, y) \in A \iff x \in U \iff f(x, y) \in U \iff (x, y) \in f^{-1}(U)$.

By part (a), f is a continuous function, so the pre-image of any open set is open. Therefore $A = f^{-1}(U)$ is open.

Exercise 6.6.

Let $x \in \overline{E}$. If $x \in E$ then $f(x) \in f(E) \subset \overline{f(E)}$, so assume $x \notin E$. Then x is a limit point of E, so there exists a sequence $\{x_n\} \in E$ such that $\lim_{n\to\infty} x_n = x$. Since f is continuous at x we have $\lim_{n\to\infty} f(x_n) = f(x)$. Since $\{x_n\} \in E$, $\{f(x_n\} \in f(E)$, so we have that $f(x) \in \overline{F(E)}$. Therefore $f(\overline{E}) \subseteq \overline{f(E)}$.
Consider the function $f(x) = e^x$ mapping from $\mathbb{R}$ to $\mathbb{R}$. Then $f(\mathbb{R}) = f(\overline{\mathbb{R}}) = (0, \infty)$. However, $\overline{f(\mathbb{R})} = \overline{(0, \infty)} = [0, \infty)$.

Exercise 6.7.

Since $\{0\}$ is closed in $\mathbb{R}$, f continuous implies that $Z(f) = f^{-1}(0)$ is closed in X (Corollary 4.8).

Exercise 6.8.

Since E is dense in X we know that $\overline{E} = X$. From Rudin exercise 2.2 we know that $f(X) = f(\overline{E}) \subseteq \overline{f(E)}$. Therefore $f(E)$ is dense in $f(X)$.
Now assume $f(x) = g(x)$ for all $x \in E$. For any $p \in X \backslash E$ we have a sequence $\{p_n\}_{n=1}^\infty \subseteq E$ that converges to p with $f(p) = \lim_{n\to\infty} f(p_n)$ and $g(p) = \lim_{n\to\infty} g(p_n)$. Since $\{p_n\}_{n=1}^\infty \subseteq E$, $f(p_n) = g(p_n)$ for all $n \in \mathbb{N}$. Therefore $f(p) = \lim_{n\to\infty} f(p_n) = \lim_{n\to\infty} g(p_n) = g(p)$. So we have $f(x) = g(x)$ for all $x \in X$.

7 June 27, 2008

Exercise 7.1.

$$x \in f^{-1}(\cup_\alpha S_\alpha) \iff f(x) \in (\cup_\alpha S_\alpha) \iff \text{there exists } \alpha' \text{ such that } f(x) \in S_{\alpha'}$$
$$\iff \text{there exists } \alpha' \text{ such that } x \in f(S_{\alpha'})$$
$$\iff x \in \cup_\alpha (S_\alpha)$$

Therefore $(\cup_\alpha S_\alpha) = \cup_\alpha (S_\alpha)$.

Exercise 7.2.
From exercise 2.7 we know that in a space with the discrete metric all sets are open. Therefore, for any function $f : X \to Y$ and any open set $U \subseteq Y$, the set $f^{-1}(U) \subseteq X$ will be open. So all functions $f : X \to Y$ is continuous.

Exercise 7.3.
We first show that h is continuous at any point $x \in [a, c]\backslash\{b\}$. Consider the case $x \in [a, b)$ (the case $x \in (b, c]$ is handled analogously). Fix any $\epsilon > 0$. Then since f is continuous at x there exists $\delta > 0$ such that for all $y \in N_\delta(x) \cap [a, b)$ we have $d(f(y), f(x)) < \epsilon$. Let $\delta' = \min\{\delta, b - x\} > 0$. Then for all $y \in N_{\delta'}(x) \cap [a, c] = N_\delta(x) \cap [a, b)$ we have $d(h(y), h(x)) = d(f(y), f(x)) < \epsilon$. So h is continuous at x.
We now consider continuity at the point b. Fix any $\epsilon > 0$. Since $\lim_{x\to b} f(x) = g(b)$ there exists $\delta_1 > 0$ such that for all $y \in N_{\delta_1}(b) \cap [a, b)$ we have $d(f(y), g(b)) < \epsilon$. Since g is continuous at b there exists $\delta_2 > 0$ such that for all $y \in N_{\delta_2}(b)\cap[b, c]$ we have $d(g(y), g(b)) < \epsilon$. Let $\delta = \min\{\delta_1, \delta_2\} > 0$. Then for all $y \in N_\delta(b)\cap[a, c]$ we have $d(h(y), h(b)) = d(h(y), g(b)) < \epsilon$. So h is continuous at b.
In general, let A, B be two disjoint sets in a metric space. Let f be a continuous function on A and g be a continuous function on B. For all points $p \in B$ that are limit points of A, let $\lim_{x\to p} f(x) = g(p)$. Similarly, for all points $q \in A$ that are limit points of B, let $\lim_{x\to q} g(x) = f(q)$. Define

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

Then h is continuous on $A \cup B$.

Exercise 7.4.
$\Rightarrow$ The forward direction of this proof follows by exercise 6.6 (Rudin exercise 4.2).
$\Leftarrow$ Proof by contrapositive. Assume that $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is not continuous. Then there exists a closed set $U \subseteq \mathbb{R}^m$ such that $f^{-1}(U)$ is not closed in A. So there exists a point $x \in (\overline{f^{-1}(U)} \cap A)\backslash f^{-1}(U)$. Since $x \notin f^{-1}(U)$ we know $f(x) \notin U$. But $f(f^{-1}(U)) \subseteq U$, and since U is closed $\overline{f(f^{-1}(U))} \subseteq U$. So $f(x) \notin \overline{f(f^{-1}(U))}$, which shows $f(\overline{f^{-1}(U)} \cap A) \not\subseteq \overline{f(f^{-1}(U))}$.

Exercise 7.5.

Note that the set $K = \{x \in \mathbb{R}^n : \alpha \leq f(x) \leq \beta\} = \{x \in \mathbb{R}^n : f(x) \in [\alpha, \beta]\} = f^{-1}([\alpha, \beta])$. Since $[\alpha, \beta]$ is closed and f is continuous, $f^{-1}([\alpha, \beta])$ is closed in $\mathbb{R}^n$. Therefore K is closed in $\mathbb{R}^n$.

Exercise 7.6.

(a) $f^{-1}(K)$ need not be connected. As a counterexample, consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ and $K = \{1\} \subseteq \mathbb{R}$. Then $f^{-1}(K) = \{-1, 1\}$, which is not connected.

(b) Let $B \subseteq \mathbb{R}^n$ be a bounded set. Then there exists $M > 0$ such that $B \subseteq \overline{N_M(0)}$. Since $\overline{N_M(0)}$ is a closed and bounded set in $\mathbb{R}^n$ it is compact (Heine-Borel). Since f is continuous, $f(\overline{N_M(0)}) \subseteq \mathbb{R}^m$ is also compact (Rudin 4.14). In $\mathbb{R}^m$ compact sets are bounded, so $f(\overline{N_M(0)})$ is bounded (Heine-Borel). Then since $f(B) \subseteq f(\overline{N_M(0)})$, $f(B)$ is also bounded.

Exercise 7.7.

$\Rightarrow$ Assume that $f(U)$ is open for all open sets $U \subseteq \mathbb{R}^n$. Let $V \subseteq \mathbb{R}^n$ be a non-empty open set. Then $f(V)$ is a non-empty open set in $\mathbb{R}$. Fix any point $y \in V$. Then $f(y) \in f(V)$, so there exists $\epsilon > 0$ such that $N_\epsilon(f(y)) \subseteq f(V)$. Since f is bounded we know that $\inf_{x \in V} f(x)$ and $\sup_{x \in V} f(x)$ both exist. Therefore $\inf_{x \in V} f(x) \leq f(y) - \epsilon < f(y) < f(y) + \epsilon \leq \sup_{x \in V} f(x)$, as desired.

$\Leftarrow$ Now assume that for all nonempty open sets $V \subseteq \mathbb{R}^n$, $\inf_{x \in V} f(x) < f(y) < \sup_{x \in V} f(x)$ for all $y \in V$. Let $U \subseteq \mathbb{R}^n$ be an open set. If $U = \emptyset$ then $f(U) = \emptyset$ which is open in $\mathbb{R}$. So assume U is non-empty. Fix any $z \in U$. Since U is open there exists an $\epsilon > 0$ such that $N_\epsilon(z) \subseteq U$. Since $N_\epsilon(z)$ is open we know that $\inf_{x \in N_\epsilon(z)} f(x) < f(z) < \sup_{x \in N_\epsilon(z)} f(x)$. So there exist $a, b \in f(N_\epsilon(z))$ such that $a < f(z) < b$. Since $N_\epsilon(z)$ is connected and f is continuous we know that $f(N_\epsilon(z))$ is also connected (Rudin 4.22). If any point $c \in (a, b)$ were not in $f(N_\epsilon(z))$, then $(-\infty, c) \cap f(N_\epsilon(z))$ and $(c, \infty) \cap f(N_\epsilon(z))$ would separate $f(N_\epsilon(z))$. Therefore $[a, b] \subseteq f(N_\epsilon(z))$. Then letting $\delta = \min\{b - f(x), f(x) - a\} > 0$ we have that $N_\delta(f(z)) \subseteq [a, b] \subseteq f(N_\epsilon(z)) \subseteq f(U)$. Therefore $f(U)$ is open, as desired.

Exercise 7.8.

Proof by contrapositive. Let $f : [0, 1] \to \mathbb{R}$ be continuous with $f(0) < 0$ and $f(1) > 0$. Let $K = \{x \in [0, 1] : f(x) > 0\}$. Fix any point $c \in [0, 1]$ such that $f(c) \neq 0$.

First assume $f(c) > 0$. So $c \neq 0$. Let $\epsilon = f(c)$. Then there exists $\delta > 0$ such that for all $x \in N_\delta(c)$, $f(x) \in (f(c) - \epsilon, f(c) + \epsilon) = (0, 2f(c))$. Therefore $N_\delta(c) \subseteq K$. Since $c \neq 0$, there exists $y \in N_\delta(c)$ such that $0 < y < c$. So $\inf K \leq y < c$. Therefore $c \neq \inf K$.

Now consider the case $f(c) < 0$. Then $c \neq 1$. Let $\epsilon = f(c)$. Then there exists $\delta > 0$ such that for all $x \in N_\delta(c)$, $f(x) \in (f(c) - \epsilon, f(c) + \epsilon) = (-2f(c), 0)$. Since $c \neq 1$, there exists $z \in N_\delta(c)$ such that $c < z < 1$. Then $[c, z] \subseteq [0, 1] \backslash K$. Therefore $c \neq \inf K$

So if $f(c) \neq 0$ we have that $c \neq \inf K$, completing the contrapositive argument.

Exercise 7.9.
Define a function $h : [0, \pi] \to \mathbb{R}$ by $h(x) = f(x) - f(x + \pi)$. As the difference of two continuous functions, h is continuous. $h(0) = f(0) - f(\pi) = f(2\pi) - f(\pi) = -h(\pi)$. If $h(0) = 0$ then we have $f(0) = f(\pi)$ so we are done. Otherwise we have $h(0) \neq 0$. Assume $h(0) < 0$ (the case $h(0) > 0$ can be handled analogously by considering $-h(x)$). Then $h(\pi) = -h(0) > 0$. Applying the previous exercise we know that there is a point $c \in [0, \pi]$ such that $h(c) = 0$. Then $f(c) = f(c + \pi)$, as desired.

Exercise 7.10.
Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous open map. Fix any points $a < b$. Then $[a, b]$ is a compact set in $\mathbb{R}$. Since f is continuous, there exists a point $c \in [a, b]$ such that $f(c) = \sup_{x\in[a,b]} f(x)$. So no neighborhood of $f(c)$ is contained in $f([a, b])$. Since $f((a, b)) \subseteq f([a, b])$, no neighborhood of $f(c)$ is contained in $f((a, b))$. The interval (a, b) is open, so $f((a, b))$ is open. Therefore $f(c) \notin f((a, b))$. So $c \in [a, b]\backslash(a, b) = \{a, b\}$.
By identical argument, the minimum value of f on $[a, b]$ is also obtained at either a or b.

Now assume f is not monotone.
Since f is not monotonically increasing there exist points $w < x$ such that $f(w) > f(x)$.
Since f is not monotonically decreasing there exist points $y < z$ such that $f(y) < f(z)$.
We can choose points $a < b$ such that $w, x, y, z \in (a, b)$.
First consider the case $f(a) \leq f(b)$.
If $f(a) < f(w)$, then the maximum of f on $[a, x]$ is greater than the value of f at both a and x, contradicting the above result.
If $f(a) \geq f(w)$, then $f(b) \geq f(a) \geq f(w) > f(x)$, so the minimum of f on $[w, b]$ is smaller than the value of f at both w and b, again contradicting the above result.
Now consider the the case $f(a) > f(b)$.
If $f(a) > f(y)$, then the minimum of f on $[a, z]$ is less than the value of f at both a and z, contradicting the above result.
If $f(a) \leq f(y)$, then $f(b) < f(a) \leq f(y) < f(z)$, so the maximum of f on $[y, b]$ is greater than the value of f at both y and b, again contradicting the above result.
We have a contradiction in all cases. Therefore f must be monotone.

Exercise 7.11.
Fix any point $y \in \mathbb{R}$ and any $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $1/N < \epsilon$. Fix any $m < N$. Let L_m be the largest integer such that $L_m/m < y$ and let U_m be the smallest integer such that $U_m/m > y$. Let $\delta = \min\{(y - L_m), (U_m - y)\}_{m=1}^{N-1} > 0$.
For any $x \in N_\delta(y)\backslash\{y\}$, either x is irrational or $x = m/n$ where m, n are integers without common divisors and $n \geq N$. In both cases we have $f(x) < \epsilon$. Therefore $\lim_{x\to y} f(x) = 0$ for all $y \in \mathbb{R}$.
If y is irrational then $f(y) = 0$, so by Theorem 4.6 we have that f is continuous at y. If y is rational than $f(y) > 0$, so again by Theorem 4.6 we know that f is not continuous at y. However, since $\lim_{x\to y} f(x)$ exists, the discontinuity at y is a simple discontinuity.

8 July 1, 2008

Exercise 8.1.

$\Rightarrow$ Assume $f : A \subset X \to Y$ is not uniformly continuous on A. Then there exists $\epsilon > 0$ such that for any $\delta > 0$ there exist points $x_\delta, y_\delta \in A$ such that $d_X(x_\delta, y_\delta) < \delta$ but $d_Y(f(x_\delta), f(y_\delta)) > \epsilon$. For $n \in \mathbb{N}$, let $\delta_n = 1/n > 0$. Let $x_n = x_{\delta_n}$ and $y_n = y_{\delta_n}$. Then $d_X(x_n, y_n) \leq 1/n$ and $d_Y(f(x_n), f(y_n)) > \epsilon$, as desired.
$\Leftarrow$ Now we are given sequences $\{x_n\}$ and $\{y_n\}$ such that $d_X(x_n, y_n) \leq 1/n$ and $d_Y(f(x_n), f(y_n)) > \epsilon$ for all $n \in \mathbb{N}$. Fix any $\delta > 0$. Then there exists $N \in \mathbb{N}$ such that $1/N < \delta$. Then $d_X(x_N, y_N) \leq 1/N < \delta$, but $d_Y(f(x_N), f(y_N)) > \epsilon$. Since $\delta > 0$ was arbitrary, we have that for any $\delta > 0$ there exist points $x_\delta, y_\delta \in A$ such that $d_X(x_\delta, y_\delta) \leq \delta$ but $d_Y(f(x_\delta), f(y_\delta)) > \epsilon$, so f is not uniformly continuous on A.

Exercise 8.2.

Let $f : B \to \mathbb{R}^k$ be a uniformly continuous function on a bounded set $B \subset \mathbb{R}^k$. Since f is uniformly continuous there exists $\delta > 0$ such that for any $x, y \in B$ with $|x - y| < \delta$, $|f(x) - f(y)| < 1$. By exercise 5.2 we know that since B is bounded in $\mathbb{R}^k$, B is totally bounded. So there exists a finite collection of points $\{x_i\}_{i=1}^n \subset \mathbb{R}^k$ such that $\{N_{\delta/2}(x_i)\}_{i=1}^n$ covers B. For each $i = 1, 2, \ldots, n$, fix $y_i \in N_{\delta/2}(x_i) \cap B$ (if such a y_i exists). Define $I \subseteq \{i\}_{i=1}^n$ by $i \in I \iff y_i$ exists. Then by the triangle inequality

$$B \subseteq \{N_{\delta/2}(x_i)\}_{i \in I} \subseteq \{N_\delta(y_i)\}_{i \in I}.$$

Let $M = \max\{|f(y_i)|\}_{i \in I}$. Then for any $x \in B$, there exists an y_k, $k \in I$ such that $|x - y_k| < \delta$, so $|f(x)| < |f(y_k)| + 1 \leq M + 1$. Therefore $f(B) \subseteq N_{M+1}(0)$, so f is bounded on B.
For an counterexample in the case where B is not bounded, consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$. For any $\epsilon > 0$ and any $x \in \mathbb{R}$, $f(N_\epsilon(x)) = N_\epsilon(x) = N_\epsilon(f(x))$, so f is uniformly continuous. However, $f(\mathbb{R}) = \mathbb{R}$ is not bounded.
For an alternative approach, note that since B is bounded there exists an $M > 0$ such that $B \subseteq N_M(0)$. Then $\overline{B} \subseteq \overline{N}_M(0)$ (by exercise 3.1). Since $\overline{B}$ is closed and bounded in $\mathbb{R}^k$, $\overline{B}$ is compact. Since B is dense in $\overline{B}$ and $\mathbb{R}^k$ is complete, by Rudin problem 4.11 below we can extend f to a continuous function $F : \overline{B} \to \mathbb{R}^k$ such that $F(x) = f(x)$ for all $x \in B$. By Theorem 4.16 we know that $F(\overline{B})$ is bounded, hence $f(B) = F(B) \subseteq F(\overline{B})$ is also bounded, as desired.
Note that in exercise 7.6 (b) we proved a similar result but with the assumption that the function f was defined on all of domain space $\mathbb{R}^n$. With this larger domain we only need to assume that f is continuous. In this problem, we assume that f is defined only on B, but we need the stronger assumption that f is uniformly continuous. For example, the arctangent function is continuous (but not uniformly continuous) on the bounded interval $(-\pi/2, \pi/2)$, but it maps this interval onto all of $\mathbb{R}$.

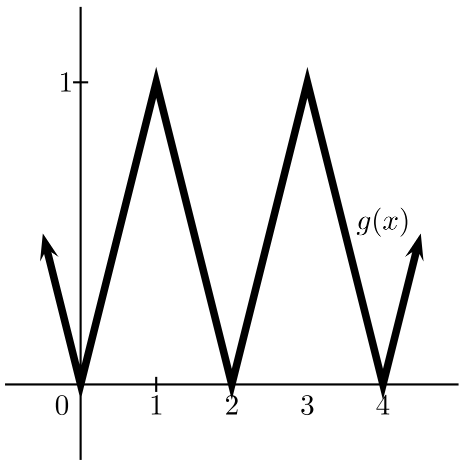
Exercise 8.3.

Let $f : X \to Y$ be a uniformly continuous function. Let $\{x_n\}$ be a Cauchy sequence in X. Fix any $\epsilon > 0$. Then there exists a $\delta > 0$ such that for $x, y \in X$ with $d_X(x, y) < \delta$, $d_Y(f(x), f(y)) < \epsilon$. Since $\{x_n\}$ is Cauchy there exists an $N \in \mathbb{N}$ such that for $m,n > N$, $d_X(x_n, x_m) < \delta$. This implies that $d_Y(f(x_n), f(x_m)) < \epsilon$. Since ϵ was arbitrary, $\{f(x_n)\}$ is Cauchy.

Exercise 8.4.
Let $f, g : \mathbb{R} \to \mathbb{R}$ be uniformly continuous. Fix any $\epsilon > 0$. Then there exists $\delta_f > 0$ such that for any $x, y \in \mathbb{R}$ with $|x-y| < \delta_f$, $|f(x) - f(y)| < \epsilon/2$. Similarly, there exists $\delta_g > 0$ such that for any $x, y \in \mathbb{R}$ with $|x-y| < \delta_g$, $|g(x) - g(y)| < \epsilon/2$. Let $\delta = \min\{\delta_f, \delta_g\}$. Then for $x, y \in \mathbb{R}$ such that $|x-y| < \delta$ we have

$$|(f+g)(x) - (f+g)(y)| \leq |f(x) - f(y)| + |g(x) - g(y)| < \epsilon/2 + \epsilon/2 = \epsilon$$

Therefore $f+g$ is uniformly continuous on $\mathbb{R}$.



For an counterexample for the product case, let $f(x) = x$ and define $g(x)$ by

$$g(x) = \begin{cases} x - \llcorner x \lrcorner & \llcorner x \lrcorner \text{ even} \\ -x + \llcorner x \lrcorner + 1 & \llcorner x \lrcorner \text{ odd} \end{cases}$$

Then for all $x, y \in \mathbb{R}$, $|f(x) - f(y)| < |x-y|$, so $g(x)$ is uniformly continuous. Let $x_n = 2n$ and let $y_n = 2n + 1/n$. Then $|x_n - y_n| = 1/n$, but $|fg(x_n) - fg(y_n)| = |2n \cdot 0 - (2n + 1/n) \cdot (1/n)| = 2 + 1/n^2 > 2$. So by exercise 8.1, fg is not uniformly continuous.

Exercise 8.5.
Part (a) is a special case of part (b) and uses the same method of proof just in a special set of circumstances. Therefore, we will provide a solution for part (b) only. The result about Cauchy sequences is exercise 8.3 above, so we only need to prove the result about extension of continuous functions (Rudin 4.13). We will show this result in the more general case where the range is any complete metric space Y.

Let E be a dense subset of a metric space X and let f be a uniformly continuous function from E to a complete metric space Y. Let $p \in X \backslash E$. Then since E is dense in X there exists a sequence $\{p_n\} \subseteq E$ that converges to p. Therefore $\{p_n\}$ is Cauchy, so by exercise 8.3, $\{f(p_n)\}$ is Cauchy in Y. Since Y is complete, $\{f(p_n)\}$ converges in Y. Define $f(p)$ to be equal to the limit of $\{f(p_n)\}$.

I claim that the value $f(p)$ is independent of the choice of sequence $\{p_n\}$. Let $\{q_n\}$ be some other sequence in E that converges to p and let $q = \lim_{n\to\infty} f(q_n)$. Fix any $\epsilon > 0$. Then there exists $\delta > 0$ such that for $x, y \in X$ with $d_X(x,y) < \delta$, $d_Y(f(x), f(y)) < \epsilon/2$. By the definition of convergence there exist $N_1, N_2, N_3, N_4 \in \mathbb{N}$ such that

$$\text{for } n > N_1, \quad d_X(p_n, p) < \delta/2, \qquad \text{for } n > N_2, \quad d_X(q_n, p) < \delta/2,$$

$$\text{for } n > N_3, \quad d_Y(f(p_n), f(p)) < \epsilon/4, \qquad \text{for } n > N_4, \quad d_X(f(q_n), q) < \epsilon/4.$$

Then for $n > \max\{N_1, N_2, N_3, N_4\}$, $d_X(p_n, q_n) < \delta$, so $d_Y(f(q_n), f(p_n)) < \epsilon/2$. Therefore

$$d_Y(q, f(p)) \leq d_Y(q, f(q_n)) + d_Y(f(q_n), f(p_n)) + d_Y(f(p_n), f(p)) < \epsilon/4 + \epsilon/2 + \epsilon/4 = \epsilon$$

Since $\epsilon > 0$ was arbitrary, $f(p) = q$. Therefore $f(p)$ is the limit of $\{f(p_n)\}$ for any sequence $\{p_n\}$ that converges to p.

So we have a well-defined function $f : X \to Y$. We want to show that f is continuous on X. Note that by our construction of f, for any $p \in X\backslash E$ and any $\epsilon, \delta > 0$, by taking n sufficiently large in any sequence $\{p_n\} \subseteq E$ that converges to p we can find a point $p_n \in N_\delta(p)$ such that $d_Y(f(p), f(p_n)) < \epsilon$.

Fix any point $p \in X$ and any $\epsilon > 0$. Then by uniform continuity we know that there exists a $\delta > 0$ such that for any $x, y \in E$ with $d_X(x, y) < \delta$ we have $d_Y(f(x), f(y)) < \epsilon/2$. By the construction of $f(p)$ we know that there is an $x \in E$ such that $x \in N_{\delta/2}(p)$ and $d_Y(f(p), f(x)) < \epsilon/4$ (if $p \in E$, just let $x = p$).

- For $y \in N_{\delta/2}(p) \cap E$ we have $d_Y(f(p), f(y)) \leq d_Y(f(p), f(x)) + d_Y(f(x), f(y)) < \epsilon/4 + \epsilon/2 < \epsilon$.
- For any $q \in N_{\delta/2}(p)\backslash E$, by our construction of $f(q)$ we know that there exists $z \in N_{\delta/2}(p)$ such that $d_Y(f(q), f(z)) < \epsilon/4$. By the triangle inequality we then have $d_Y(f(p), f(q)) \leq d_Y(f(p), f(x)) + d_Y(f(x), f(z)) + d_Y(f(z), f(q)) < \epsilon/4 + \epsilon/2 + \epsilon/4 < \epsilon$.

So $f(N_{\delta/2}(p)) \subseteq N_\epsilon(f(p))$. Since ϵ was arbitrary, f is continuous at p. Therefore f is continuous on X, as desired. The idea of continuous extensions is useful in a number of contexts, for example Rudin problem 4.8 (exercise 8.2).

Exercise 8.6.

Let f be a continuous mapping of a compact metric space X into a metric space Y. Assume f is not uniformly continuous. Then by exercise 8.1 there exist sequences $\{p_n\}, \{q_n\} \subseteq X$ such that $d_X(p_n, q_n) < 1/n$ but $d_Y(f(p_n), f(q_n)) \geq \epsilon$.

Since X is compact, by Theorem 3.6 we know that $\{p_n\}_{n=1}^\infty$ has a convergent subsequence $\{p_{n_i}\}_{i=1}^\infty$. Let $p \in X$ be the limit of this subsequence. Then I claim that $\{q_{n_i}\}_{n=1}^\infty$ also converges to p.

Fix any $\epsilon > 0$. Then there exists $N_1 \in \mathbb{N}$ such that for all $i \geq N_1$, $d_x(p_{n_i}, p) < \epsilon/2$. There also exists $N_2 \in \mathbb{N}$ such that for all $n > N_2$, $d_X(p_n, q_n) < \epsilon/2$. Since $n_i \geq i$ for all $i \in \mathbb{N}$, for $i > \max\{N_1, N_2\}$ we have by the triangle inequality $d_X(p, q_{n_i}) < \epsilon$. Since ϵ was arbitrary, this shows that $\{q_{n_i}\}_{n=1}^\infty$ converges to p.

Since f is continuous, by Theorem 4.2 we have that $\lim_{i\to\infty} f(p_{n_i}) = f(p) = \lim_{i\to\infty} f(q_{n_i})$. So there exist $M_1, M_2 \in \mathbb{N}$ such that for $i > M_1$, $d_Y(f(p), f(p_{n_i})) < \epsilon/2$ and for $i > M_2$, $d_Y(f(p), f(q_{n_i})) < \epsilon/2$. Then for $i > \max\{M_1, M_2\}$, $d_Y(f(p_{n_i}), f(q_{n_i})) < \epsilon$, contradicting our construction of the points p_n and q_n.

Therefore f must be uniformly continuous.

Exercise 8.7.

Let f be a continuous mapping of $I = [0, 1]$ into itself. Define a new function $h : I \to \mathbb{R}$ by $h(x) = f(x) - x$. Since f is continuous and $g(x) = -x$ is continuous, by Theorem 4.9 we know that h is continuous. Since

$f(I) \subseteq I$, $h(0) = f(0) \in I$. If $h(0) = 0$ then $f(0) = 0$ and we are done, so assume $h(x) \neq 0$, so $h(0) > 0$. Similarly, $h(1) = f(1) - 1 \in [-1, 0]$. If $h(1) = 0$ then $f(1) = 1$ and we are done, so assume $h(x) \neq 0$, so $h(1) < 0$. Then by Theorem 4.23 there must be some $x \in (0, 1)$ such that $h(x) = 0$, so $f(x) = x$, as desired.

Exercise 8.8.

(a) Fix any two points $x, y \in X$ any any non-empty set $A \subset X$. Fix any point $z \in A$. Then

$$f_A(x) = \inf_{p \in A} d(x, p) \leq d(x, z) \leq d(x, y) + d(y, z).$$

Since this holds for all $z \in A$ we have

$$f_A(x) \leq d(x, y) + \inf_{p \in A} d(y, p) = d(x, y) + f_A(y).$$

Therefore $f_A(x) - f_A(y) \leq d(x, y)$. By identical argument with the roles of x and y reversed we obtain $f_A(y) - f_A(x) \leq d(x, y)$. Therefore $|f_A(x) - f_A(y)| \leq d(x, y)$.

Therefore, for any $\epsilon > 0$ and any $x, y \in X$, if $d(x, y) < \epsilon$ then $|f_A(x) - f_A(y)| < \epsilon$. So f_A is a uniformly continuous function on X.

(b) $\Rightarrow$ Assume that $x \in \overline{A}$. Then either $x \in A$ or x is a limit point of A. If $x \in A$ then since $d(x, x) = 0$ we have $f_A(x) = 0$. If x is a limit point of A then for every $\epsilon > 0$ we know that there exists $z \in A$ such that $d(x, z) < \epsilon$, therefore $f_A(x) = \inf_{z \in A} d(x, z) = 0$.

$\Leftarrow$ Assume that $f_A(x) = 0$. By the definition of $f_A(x)$ we have $\inf_{z \in A} d(x, z) = 0$. So either there exists a $z \in A$ such that $d(z, x) = 0$ or for each $n \in N$ there exists $z_n \in A$ such that $0 < d(x, z_n) \leq 1/n$. In the former case, since d is a metric we have $x = z$, so $x \in A \subseteq \overline{A}$. In the later case, $\{z_n\}$ is a sequence in A converging to x, so x is in $\overline{A}$.

Exercise 8.9.

Let K and F be disjoint sets in a metric space X, K is compact, F is closed. From the previous exercise we know that $f_F(x)$ is a continuous function on X. Since $K \subseteq X$ is compact, we know that there exists a point $k \in K$ such that $f_F(k) = \inf_{x \in K} f_F(x)$. Since F is closed and $K \cap F = \emptyset$, $k \notin \overline{F} = F$. So by part (b) of the previous exercise, $\inf_{x \in K} f_F(x) = f_F(k) > 0$.

Let $\delta = f_F(k)/2 > 0$. Then for all $p \in K$ and $q \in F$ we have $d(p, q) \geq f_F(p) \geq f_F(k) > \delta$, as desired.

For a counterexample in the case where neither K or F is compact, let $K = \mathbb{N}$ and $F = \{n + \frac{1}{2n} : n \in \mathbb{N}\}$, both considered as subsets of $\mathbb{R}$. Both F and K have no limit points and are therefore closed. However, $d(n, n + \frac{1}{2n}) = \frac{1}{2n}$, so there are pairs of points from F and K getting arbitrarily close.

9 July 2, 2008

These are all unfinished problems from problem sets 7 and 8.

10 July 7, 2008

Exercise 10.1.

Fix any point $x \in \mathbb{R}$. Then $0 \leq |\phi(t)| = \frac{|f(t)-f(x)|}{|t-x|} \leq |t-x|$ for all $t \in \mathbb{R}$. Since $\lim_{t\to x} |t-x| = 0$ by the squeeze theorem we have $\lim_{t\to x} |\phi(t)| = 0$. Therefore $f'(x) = \lim_{t\to x} \phi(t) = 0$. So f is differentiable on all of $\mathbb{R}$ with derivative always zero, which by Rudin Theorem 5.11 (b) implies that f is constant.

Exercise 10.2.

Lemma. *Let h be a real continuous function on $[a,b]$ which is differentiable in (a,b). If $|h'(x)| \leq M$ for all $x \in (a,b)$, then $|h(a) - h(b)| \leq M(b-a)$.*

Proof. From the mean value theorem we know that there exits a point $c \in (a,b)$ such that $|h(b) - h(a)| = (b-a)|h'(c)|$. Then since $|h'(c)| \leq M$ we have the desired result. ☐

Fix any $\epsilon > 0$. Since $f'(x) \to 0$ as $x \to \infty$, there exists $N \in \mathbb{R}$ such that for $x \geq N$, $|f'(x)| < \epsilon$. So for $x \geq N$, by our lemma we know that $|g(x)| = |f(x+1) - f(x)| \leq \epsilon((x+1) - x) = \epsilon$. Therefore $g(x) \to 0$ as $x \to \infty$.

Exercise 10.3.

Fix any $\epsilon > 0$. Since f' is continuous on the compact set $[a,b]$ we know that f' is uniformly continuous on $[a,b]$. So there exists a $\delta > 0$ such that for any $y, z \in [a,b]$ with $|y-z| < \delta$ we have $|f'(y) - f'(z)| < \epsilon$. Fix any two points $t, x \in [a,b]$ with $0 < |t-x| < \delta$. From the mean value theorem there exists a point c between x and t such that $\frac{f(t)-f(x)}{t-x} = f'(c)$. Since c is between x and t we know that $|x-c| < |x-t| < \delta$, so by uniform continuity we have:

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| = |f'(c) - f'(x)| < \epsilon$$

So f is uniformly differentiable, as desired.

Exercise 10.4.

For real valued functions, this result follows immediately from L'Hospital's rule if we can show that there is a value $b > 0$ such that $g'(x) \neq 0$ for all $x \in (0,b)$. Since $g'(x) \to B \neq 0$ as $x \to 0$, there exists $\epsilon > 0$ such that for all $x \in (0,\epsilon)$, $|g'(x) - B| < |B|/2$. Then for $x \in (0,\epsilon)$ we have $|g'(x)| > |B| - |B|/2 = |B|/2 > 0$. So $g'(x) \neq 0$ on $(0,\epsilon)$, as desired.

Exercise 10.5.

Suppose f is defined in a neighborhood of x and $f''(x)$ exists. Let $g(h) = f(x+h) + f(x-h) - 2f(x)$ and let $G(h) = h^2$. Since f is differentiable at x, we know that f is continuous at x. So $\lim_{h\to 0} f(x+h) = \lim_{h\to 0} f(x-h) = f(x)$. Therefore $\lim_{h\to 0} g(h) = 0$. We also know that x^2 is continuous so $\lim_{h\to 0} h^2 = 0$. Using the chain rule, taking the derivative of g *with respect to h* yields $g'(h) = f'(x+h) - f'(x-h)$. (Note that since $f''(x)$ exists, f' exists on a neighborhood of x (see Rudin section 5.14), so we can consider $f'(x+h)$

for $h > 0$. However, f'' may only exist at the point x, so we can not consider $f''(x+h)$ for $h > 0$, which prevents us from applying L'Hospital's rule a second time.) The derivative of h^2 with respect to h is $2h$. Therefore we have

$$\begin{aligned}\lim_{h\to 0}\frac{g'(h)}{G'(h)} = \lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h} &= \lim_{h\to 0}\frac{f'(x+h)-f'(x)-(f'(x-h)-f'(x))}{2h} \\ &= \frac{1}{2}\left(\lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h} + \lim_{h\to 0}\frac{f'(x-h)-f'(x)}{-h}\right) \\ &= \frac{1}{2}\left(\lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h} + \lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h}\right) \\ &= \frac{1}{2}\left(f''(x)+f''(x)\right) = f''(x)\end{aligned}$$

In the second to last line above we used the fact that for any function $F(h)$ defined on a neighborhood of 0, $\lim_{h\to 0} F(h) = \lim_{h\to 0} F(-h)$ (assuming at least one of the limits exists). This can be proved by noting that $N_\delta(0) = \{-x : x \in N_\delta(0)\}$.

Now, applying theorem 5.13 we have

$$\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2} = \lim_{h\to 0}\frac{g(h)}{G(h)} = \lim_{h\to 0}\frac{g'(h)}{G'(h)} = f''(x)$$

as desired.

Consider the function

$$f(x) = \begin{cases} -x^2 & x < 0 \\ x^2 & x \geq 0 \end{cases}$$

then by the following exercise, f"(0) does not exist. However, using the fact that f is an odd function we have

$$\lim_{h\to 0}\frac{f(0+h)+f(0-h)-2f(0)}{h^2} = \lim_{h\to 0}\frac{0}{h^2} = 0$$

Exercise 10.6.

For $x > 0$, $f(x) = x^3$ for all $x \in N_x(x)$. So the derivatives of f are just the standard derivatives of x^3. Similarly, for $x < 0$, the derivatives of f are the standard derivatives of $-x^3$. The case $x = 0$ requires more careful attention.

For $h > 0$, $\frac{f(h)-f(0)}{h} = \frac{h^3}{h} = h^2$.

For $h < 0$, $\frac{f(h)-f(0)}{h} = \frac{-h^3}{h} = -h^2$.

So $\lim_{h\to 0}\left|\frac{f(h)-f(0)}{h}\right| = \lim_{h\to 0}|h^2| = 0$, which means $f'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = 0$.

So

$$f'(x) = \begin{cases} -3x^2 & x < 0 \\ 3x^2 & x \geq 0 \end{cases}$$

For $h > 0$, $\frac{f'(h)-f'(0)}{h} = \frac{3h^2}{h} = 3h$.

For $h < 0$, $\frac{f'(h)-f'(0)}{h} = \frac{-3h^2}{h} = -3h$.

So $\lim_{h\to 0}\left|\frac{f'(h)-f'(0)}{h}\right| = \lim_{h\to 0}|3h| = 0$, which means $f''(0) = \lim_{h\to 0}\frac{f'(h)-f'(0)}{h} = 0$.
So

$$f''(x) = |x| = \begin{cases} -6x & x < 0 \\ 6x & x \geq 0 \end{cases}$$

For $h > 0$, $\frac{f''(h)-f''(0)}{h} = \frac{6h}{h} = 6$.
For $h < 0$, $\frac{f''(h)-f''(0)}{h} = \frac{-6h}{h} = -6$.
Therefore $f'''(0) = \lim_{h\to 0}\frac{f''(h)-f''(0)}{h}$ does not exist, since every neighborhood of zero contains both points that map to 6 and points that map to -6.

11 July 9, 2008

Exercise 11.1.

Let $\{f_n\}$ be a sequence of bounded functions on E that converge uniformly to a function f. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in E$, $|f_n(x) - f(x)| \leq 1$. For all $n \in \mathbb{N}$ we know that f_n is bounded, so for all $n \in \mathbb{N}$ there exists an $M_n \in \mathbb{R}$ such that $|f_n(x)| \leq M_n$ for all $x \in E$. Then by the triangle inequality we know that $|f(x)| \leq M_N + 1$ for all $x \in E$. Applying the triangle inequality again we have that for $n \geq N$, $|f_n(x)| \geq M_N + 2$ for all $x \in E$. Let $M = \max\{M_1, M_2, \ldots, M_{N-1}, M_N + 2\}$. Then for all $x \in E$ and for all $n \in \mathbb{N}$ we have $|f_n(x)| \leq M$. Therefore $\{f_n\}$ is uniformly bounded, as desired.
Note that M is also a bound for the limit function f. This fact will be used in the following exercise.

Exercise 11.2.

Let $\{f_n\}$ converge uniformly to f on E and let $\{g_n\}$ converge uniformly to g on E. Fix any $\epsilon > 0$. Then there exist $N_1, N_2 \in \mathbb{N}$ such that for $n > N_1$, $x \in E$ we have $|f_n(x) - f(x)| \leq \epsilon/2$ and for $n > N_2$, $x \in E$ we have $|g_n(x) - g(x)| \leq \epsilon/2$. Then for $n > N = \max\{N_1, N_2\}$, by the triangle inequality we have

$$|(f_n + g_n)(x) - (f + g)(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| \leq \epsilon/2 + \epsilon/2 = \epsilon$$

for all $x \in E$. Therefore $\{f_n + g_n\}$ converges uniformly to $f + g$ on E.
Now assume additionally that $\{f_n\}$ and $\{g_n\}$ are bounded functions. By the proof of Rudin exercise 7.1, we know that there is a real number M_1 that bounds all the $\{f_n\}$ and also the limit function f. Similarly, there exists an M_2 that bounds the $\{g_n\}$ and the function g. Let $M = \max\{M_1, M_2, 1\}$. Fix any $\epsilon > 0$. Proceeding as above we know that there is an $N \in \mathbb{N}$ such that for all $n > N$ and $x \in E$ we have $|f_n(x) - f(x)| \leq \epsilon/2M$ and $|g_n(x) - g(x)| \leq \epsilon/2M$. Then for all $n > N$ and $x \in E$ we have

$$\begin{aligned}
|f_n(x)g_n(x) - f(x)g(x)| &\leq |f_n(x)g_n(x) - f_n(x)g(x)| + |f_n(x)g(x) - f(x)g(x)| \\
&= |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| \\
&\leq M \cdot \frac{\epsilon}{2M} + M \cdot \frac{\epsilon}{2M} \\
&= \epsilon.
\end{aligned}$$

Therefore $\{f_n g_n\}$ converges uniformly to fg on E.

Exercise 11.3.

(a) We first show that $\{f_n\}$ converges uniformly to $f(x) = x$ on any bounded interval. Fix any bounded interval $[c, d]$ and any $\epsilon > 0$. Let $M > \max\{|c|, |d|\}$ be a bound on our interval. Fix $N \in \mathbb{N}$ such that $N > M/\epsilon$. Then for $n > N$ and $x \in [c, d]$ we have

$$|f_n(x) - f(x)| = \left| x\left(1 + \frac{1}{n}\right) - x \right| = |x| \cdot \left|\frac{1}{n}\right| \leq M \cdot \frac{\epsilon}{M} = \epsilon$$

Therefore $\{f_n\}$ converges uniformly to f on $[c, d]$.

Define the function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 0 & \text{if either } x = 0 \text{ or } x \notin \mathbb{Q} \\ b & \text{if } x \in Q \text{ and } x = \frac{a}{b}, b > 0,\ a, b \text{ relatively prime.} \end{cases}$$

We claim that $\{g_n\}$ converges uniformly to g on any bounded interval (in fact, the convergence is also uniform on all of $\mathbb{R}$). Fix any bounded interval $[c, d]$ and any $\epsilon > 0$. Fix $N \in \mathbb{N}$ such that $N > 1/\epsilon$. Notice that for all points in $\mathbb{R}$, $|g_n(x) - g(x)| = 1/n$. So for $n > N$ and $x \in [c, d]$ we have $|g_n(x) - g(x)| = \frac{1}{n} \leq \epsilon$. Therefore $\{f_n\}$ converges uniformly to f on $[c, d]$.

(b) Fix any bounded interval (c, d). We first show that for any $M > 0$ there exists a point $\frac{a}{b} \in Q \cap (c, d)$ such that $b > 0$, a, b are relatively prime, and $|a| > M$. Since there are an infinite number of primes, we can choose a prime p such that $\frac{5}{p} < (d - c)$. Let $q - 1$ be the smallest integer such that $\frac{q-1}{p} \in (c, d)$ but $q \neq 0$. Then $\frac{q}{p}$ and $\frac{q+1}{p}$ are also in (c, d). Let $k \in \mathbb{N}$ such that $p^k > M/|q|$. Assume $q > 0$. Let $x := \frac{qp^k+1}{p^{k+1}}$. Then $c < \frac{q}{p} < \frac{qp^k+1}{p^{k+1}} < \frac{q+1}{p} < d$, so $x \in (c, d)$. Since qp^k is a multiple of p, $qp^k + 1$ and p^k are relatively prime. Finally, $|qp^k + 1| > |qp^k| > |q|\frac{M}{|q|} = M$, so x satisfies the desired properties. If $q < 0$, considering $x := \frac{qp^k-1}{p^{k+1}}$ yields the desired result.

Let $h(x) = f(x)g(x)$. Then we know from Theorem 3.3 that $\{h_n\}$ converges point-wise to h. Therefore the only possible function to which $\{h_n\}$ could converge uniformly is h. We will show that this does not occur.

Fix any $N \in \mathbb{N}$. Then by the above work we know there exists a point $x = \frac{a}{b} \in (c, d) \cap \mathbb{Q}$ with $b > 0$, a, b relatively prime, and $|a| > N$. Then

$$|h_N(x) - h(x)| = \left| \frac{a}{b} \left(1 + \frac{1}{N}\right) \left(b + \frac{1}{N}\right) - \frac{a}{b} \cdot b \right| = |a| \cdot \left| \frac{1}{N} + \frac{1}{bN} + \frac{1}{N^2} \right| \geq \frac{|a|}{N} \geq 1$$

Therefore $\{h_n\}$ does not converge uniformly to h on (c, d), so $\{h_n\}$ does not converge uniformly on any bounded interval.

Exercise 11.4.

(a) We first show that $\{f_n\}$ converges point-wise to the function

$$f(x) = \begin{cases} 0 & x \in [0, 1) \\ 1 & x = 1 \end{cases}$$

Point-wise convergence at $x = 0$ and $x = 1$ follows from the fact that for all $n \in \mathbb{N}$, $f_n(0) = 0$ and $f_n(1) = 1$. For $x \in (0, 1)$, from HW 4.4 (a) we know that $\lim_{n\to\infty} x^n = 0$, so $f_n(x)$ converges to $f(x)$.

If $\{f_n\}$ were to converge uniformly on $[0, 1]$ it would have to converge to its point-wise limit f. However, each of the functions $\{f_n\}$ are continuous, while the function f is not continuous at $x = 1$. By theorem 7.12, the uniform limit of continuous functions is continuous, so $\{f_n\}$ can not converge uniformly on [0,1].

(b) We will show that $\{gf_n\}$ converges uniformly to the zero function on $[0, 1]$. Fix any $\epsilon > 0$. Since g is continuous at 1 there exists a $\delta > 0$ such that for $x \in (1 - \delta, 1]$, $|g(x)| = |g(x) - g(1)| < \epsilon$. Then for all $x \in (1 - \delta, 1]$ and all $n \in \mathbb{N}$ since $|f_n(x)| \leq 1$, we have $|gf_n(x)| \leq |g(x)| < \epsilon$.

Since g is a continuous function on the compact set $[0, 1]$, g is bounded. Let $M > 0$ be a bound for g. Let $y \in (1 - \delta, 1)$. Then from part (a) $f_n(y)$ converges to zero. So there exists $N \in \mathbb{N}$ such that for $n > N$, $|f_n(y)| < \epsilon/M$. Since the f_n are increasing, non-negative functions, for $n > N$ and $x \in [0, y]$ we have $|f_n(x)| < \epsilon/M$. So for $n > N$ and $x \in [0, y]$, $|gf_n(x)| < M \cdot \epsilon/M = \epsilon$.

Combining the above work we have for all $x \in [0, 1]$ and all $n > N$, $|gf_n(x)| < \epsilon$. Therefore $\{gf_n\}$ converges uniformly to the zero function on $[0, 1]$.

An alternative approach is to show that $\{gf_n\}$ converges point-wise to the zero function on $[0, 1]$ and then to apply Theorem 7.13.

Exercise 11.5.

(a) B is not open. Consider the function $f \in B$ defined by

$$f(x) = \begin{cases} 1 & x \in (-\infty, 1] \\ \frac{1}{x} & x \in (1, \infty) \end{cases}$$

Fix any $\epsilon > 0$. Define a new function $g \in C_b(\mathbb{R})$ by $g(x) = f(x) - \epsilon/2$. Then for $x > 2/\epsilon$ we have $g(x) = f(x) - \epsilon/2 < \epsilon/2 - \epsilon/2 = 0$, so $g \notin B$. However, $||f - g|| = \epsilon/2 < \epsilon$. So every ϵ neighborhood of f contains elements not in B. Therefore B is not open.

(b) $\text{Int}(B) = \{f(x) \in C_b(\mathbb{R}) : \inf_{x\in\mathbb{R}} f(x) > 0\}$.

$\subseteq$ If $f \in B$ but $\inf_{x\in\mathbb{R}} f(x)$ is not strictly positive, then we must have $\inf_{x\in\mathbb{R}} f(x) = 0$. Fix any $\epsilon > 0$. Then there exists a point $p \in \mathbb{R}$ such that $f(p) < \epsilon/2$. So the function $g(x) = f(x) - \epsilon/2$ is not in B but $g \in N_\epsilon(f)$. Therefore f is not an interior point.

$\supseteq$ Let $f \in C_b(\mathbb{R})$ with $\inf_{x\in\mathbb{R}} f(x) = c > 0$. Let $\epsilon = c/2 > 0$. Then if $g \in N_\epsilon(f)$, then for all $x \in \mathbb{R}$ we have $g(x) > f(x) - \epsilon > c - c/2 = c/2 > 0$, so $g \in B$. Therefore f is an interior point of B.

(c) $\overline{B} = \{f(x) \in C_b(\mathbb{R}) : f(x) \geq 0 \quad \forall x \in \mathbb{R}\}$.

$\subseteq$ Let $f \in C_b(\mathbb{R})$ such that $f(p) < 0$ for some $p \in \mathbb{R}$. Then for any $g \in B$, $g(p) > 0$, so $|g(p) - f(p)| > |f(p)|$. Therefore $||g - f|| > |f(p)|$ for all $g \in B$, so $N_{|f(p)|}(f) \cap B = \emptyset$ and $f \notin \overline{B}$.

$\supseteq$ Let $f \in \{f(x) \in C_b(\mathbb{R}) : f(x) \geq 0 \quad \forall x \in \mathbb{R}\}$. Fix any $\epsilon > 0$. Consider the function g defined by $g(x) = f(x) + \epsilon/2$ for all $x \in \mathbb{R}$. Then $g \in B$ and $||f - g|| = \epsilon/2$. Since this holds for all $\epsilon > 0$, f is a limit point of B, so $f \in \overline{B}$.

Exercise 11.6.

Since uniform convergence implies point-wise convergence, for each point $x \in K$, $f(x)$ is the limit of the sequence $\{f_n(x)\}$. For all $n \in \mathbb{N}$, $|f_n(x)| \leq M$, therefore $f(x) \leq M$. So $f(x) \in \overline{N_M(0)}$ for all $x \in K$.

Fix any $\epsilon > 0$. Since g is continuous on the compact set $\overline{N_M(0)}$, g is uniformly continuous on $\overline{N_M(0)}$. So there exists $\delta > 0$ such that for all $x, y \in \overline{N_M(0)}$ with $|x - y| < \delta$ we have $|g(x) - g(y)| < \epsilon$.

Since $f_n \to f$ uniformly on K there exists $N \in \mathbb{N}$ such that for all $n > N$ and all $x \in K$, $|f_n(x) - f(x)| < \delta$. Therefore, for all $x \in K$ and all $n > N$ we have $|h_n(x) - h(x)| = |g[f_n(x)] - g[f(x)]| < \epsilon$. Therefore $h_n \to h$ uniformly on K, as desired.

Exercise 11.7.

(a) Fix $x \in (0, 1)$. Fix any $\epsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon x}$. Then for $n > N$ we have $|f_n(x)| = \frac{1}{nx+1} < \frac{1}{Nx+1} < \frac{1}{\frac{1}{\epsilon}+1} < \frac{1}{\frac{1}{\epsilon}} = \epsilon$. Therefore $|f_n(x)|$ converges to zero, so f_n converges point-wise to the zero function on $(0, 1)$.

To show that this convergence is not uniform, consider $\epsilon = 1/2$ and fix any $N \in \mathbb{N}$. Then there exists $y \in (0, 1)$ such that $y < 1/N$. Then $|f_N(y)| = \frac{1}{yN+1} > \frac{1}{1+1} = \epsilon$. Therefore f_n does not converge uniformly to the zero function on $(0, 1)$.

(b) Fix any $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $N > 1/\epsilon$. Then for $n > N$ and $x \in (0, 1)$ we have $|g_n(x)| = \frac{1}{n+1/x} < \frac{1}{n} < \frac{1}{N} < \epsilon$. Therefore g_n converges uniformly to the zero function on $(0, 1)$.

Exercise 11.8.

Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Fix any point $x \in E$ and let $\{x_n\}$ be any sequence of points in E that converges to x. Fix any $\epsilon > 0$. By Theorem 7.12, f is continuous at x. Therefore $\lim_{n\to\infty} f(x_n) = f(x)$ (Theorems 4.6 and 4.2). So there exists an $N_1 \in \mathbb{N}$ such that for all $n > N_1$, $d(f(x_n), f(x)) < \epsilon/2$. Also, from uniform convergence we know that there exists an $N_2 \in \mathbb{N}$ such that for all $n > N_2$ and all $y \in E$, $|f_n(y) - f(y)| \leq \epsilon/2$. Let $N = \max\{N_1, N_2\}$. Then for $n > N$ we have $|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \leq \epsilon/2 + \epsilon/2 = \epsilon$. Therefore $\lim_{n\to\infty} f_n(x_n) = f(x)$, as desired.

The converse is not true. Consider the functions $f_n(x) = x/n$ on $\mathbb{R}$. Let f be the zero function on $\mathbb{R}$. Fix any $x \in R$ and let $\{x_n\}$ be any sequence that converges to x. Fix any $\epsilon > 0$. Since $\{x_n\}$ converges to x there exists $N \in \mathbb{N}$ such that for all $n > N$, $|x_n - x| < 1$. Then for $n > N$ we have $|f_n(x_n)| \leq \frac{|x|+1}{n}$. So for $n > \frac{1}{\epsilon} \cdot \max\{N, |x| + 1\}$ we have $|f_n(x_n)| \leq \frac{|x|+1}{n} \leq \epsilon$. Since ϵ was arbitrary, this implies that $\lim_{n\to\infty} f_n(x_n) = 0 = f(x)$. However, $\{f_n\}$ does not converge to f uniformly on $\mathbb{R}$.

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Exercise 12.1.

This is exercise 11.8.

Exercise 12.2.

(a) Fix any $\delta > 0$. Then there exists $N \in \mathbb{N}$ such that $N > 1/\delta$. Then $|0 - \frac{1}{N}| = \frac{1}{N} < \delta$ and $|f_N(0) - f_N(\frac{1}{N})| = |0 - 1| = 1$. Therefore $\{f_n : n \in \mathbb{N}\}$ is not equicontinuous.

(b) Note that each of the f_n from part (a) are in S. Therefore, we can show that S is not compact by showing that $\{f_n\}$ has no convergent subsequence.

Let $\{f_{n_i}\}$ be any subsequence of $\{f_n\}$. For any point $x \in (0, 1]$, $f_{n_i}(x) = 0$ for $n_i > 2/x$. Additionally, $f_{n_i}(0) = 0$ for all $i \in \mathbb{N}$ Therefore f_{n_i} converges point-wise to the zero function on [0,1].

So to show that $\{f_{n_i}\}$ has no uniform limit on [0,1] we need only show that it does not converge uniformly to the zero function. But for all $i \in \mathbb{N}$, $|f_{n_i}(\frac{1}{n_i})| = 1$. Therefore, the arbitrary subsequence $\{f_{n_i}\}$ has no uniform limit on [0,1], so S is not compact.

For an alternative proof, note that the argument of part (a) can also be applied to any subsequence $\{f_{n_i}\}$ of $\{f_n\}$ to show that all subsequences are not equicontinuous. Then by Theorem 7.24 we know that $\{f_{n_i}\}$ has no uniform limit, which implies S is not compact.

Exercise 12.3.

(a) Note that by the chain rule $f_n'(x) = \cos \frac{x}{n}$. Therefore $|f_n'(x)| \leq 1$ for all $x \in [0, \infty)$. Now fix any $\epsilon > 0$. Let $x, y \in [0, \infty)$ such that $|x - y| < \epsilon$ Then by the lemma of exercise 10.2 we know that $|f_n(x) - f_n(y)| \leq |x - y| < \epsilon$ for all $n \in \mathbb{N}$. Therefore $\mathbb{F}$ is equicontinuous.

(b) Let $M > 0$ be a uniform bound for $\mathbb{F}'$. Fix any $\epsilon > 0$. Let $x, y \in [a, b]$ such that $|x - y| < \epsilon/M$. Then for any $f \in \mathbb{F}$, by the lemma of exercise 10.2 we know that $|f(x) - f(y)| \leq M|x - y| < \epsilon$. Therefore $\mathbb{F}$ is equicontinuous.

Exercise 12.4.

Fix any $\epsilon > 0$. Since $\{f_n\}$ is equicontinuous there exists $\delta > 0$ such that for all $x, y \in K$ with $d(x, y) < \delta$ and for all $n \in \mathbb{N}$ we have $|f_n(x) - f_n(y)| < \epsilon/3$. Since K is compact it is also totally bounded, so there exist points $\{x_i\}_{i=1}^m \subseteq K$ such that $K \subseteq \cup_{i=1}^m N_\delta(x_i)$. Since $\{f_n\}$ converges point-wise, for each $i \in \{1, 2, \ldots, m\}$ the sequence $\{f_n(x_i)\}$ is Cauchy. So for each $i \in \{1, 2, \ldots, m\}$ there exists N_i such that for all $n, m > N_i$, $|f_n(x_i) - f_m(x_i)| < \epsilon/3$. Let $N = \max\{N_i\}_{i=1}^m$.

Now fix any point $x \in K$. Then there is some $k \in \{1, 2, \ldots, m\}$ such that $x \in N_\delta(x_k)$. Therefore, for $n, m > N$ we have

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(x_i)| + |f_n(x_i) - f_m(x_i)| + |f_m(x_i) - f_m(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

So by Theorem 7.8, $\{f_n\}$ converges uniformly on K.

Exercise 12.5.

Since all compact sets in any metric space are closed and bounded, S is closed and bounded in $C(K)$ (which also implies that the functions in S are uniformly bounded).

Fix any $\epsilon > 0$. Since S is compact it is totally bounded. So there exist functions $\{f_i\}_{i=1}^m \subseteq S$ such that $S \subseteq \cup_{i=1}^m N_{\epsilon/3}(f_i)$. Each f_i is a continuous function on the compact set K and hence is uniformly continuous. So for each $i \in \{1, 2, \ldots, m\}$ there exists a δ_i such that for all points $x, y \in K$ with $d(x, y) < \delta_i$ we have $|f_i(x) - f_i(y)| < \epsilon/3$. Let $\delta = \min\{\delta_i\}_{i=1}^m$.

Now fix any $f \in S$. Then there is some $k \in \{1, 2, \ldots, m\}$ such that $f \in N_{\epsilon/3}(f_k)$. So for all $z \in K$ we have $|f(z) - f_k(z)| < \epsilon/3$. So for any $x, y \in K$ with $d(x, y) < \delta$ we have

$$|f(x) - f(y)| \leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

Therefore S is equicontinuous.

An alternative approach is to prove the contrapositive of the result. Assume S were not equicontinuous. Then there exists $\epsilon > 0$ such that for all $\delta > 0$ there are points $x, y \in K$ and a function $f \in S$ such that $d(x, y) < \delta$ but $|f(x) - f(y)| \geq \epsilon$. So for each $n \in \mathbb{N}$ we can pick $f_n \in S$ and $x_n, y_n \in K$ such that $d(x_n, y_n) < 1/n$ but $|f_n(x_n) - f_n(y_n)| \geq \epsilon$. Then since any subsequence of $\{f_n\}$ contains terms with arbitrarily large n, no subsequence of $\{f_n\}$ is equicontinuous. So by Theorem 7.24, no subsequence of $\{f_n\}$ converges in $C(K)$. Therefore S is not compact, proving the contrapositive of the desired result.

Exercise 12.6.

(a) Let $\{f_n\}$ be a sequence of monotonically increasing functions on $\mathbb{R}$ with $0 \leq f_n(x) \leq 1$ for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$. Since $\mathbb{Q}$ is countable and the $\{f_n\}$ are point-wise bounded, by Theorem 7.23 we know that there exists a sequence $\{n_k\}$ such that $\{f_{n_k}(r)\}$ converges for all $r \in \mathbb{Q}$. Define $g(r) : \mathbb{Q} \to \mathbb{R}$ by letting $g(r)$ be the limit of $\{f_{n_k}(r)\}$ for all $r \in \mathbb{Q}$.

Note that since the f_n are all monotone increasing, for $x, y \in \mathbb{Q}$, $x < y$ we have $g(x) = \lim_{k\to\infty} f_{n_k}(x) \leq \lim_{k\to\infty} f_{n_k}(y) = g(y)$, so $g(r)$ is a non-decreasing function. Define a function f on $\mathbb{R}$ by letting $f(x) = \sup\{g(r) : r \in \mathbb{Q}, r \leq x\}$. Since g is non-decreasing on $\mathbb{Q}$, for $r \in \mathbb{Q}$ we have $f(r) = g(r)$, so $\lim_{k\to\infty} f_{n_k}(r) = f(r)$ for $r \in \mathbb{Q}$. Note that from its definition the new function f is non-decreasing on $\mathbb{R}$.

Let $x \in \mathbb{R}$ be any point at which f is continuous. Fix any $\epsilon > 0$. Then there exists a $\delta > 0$ such that for all $y \in N_\delta(x)$, $f(y) \in N_{\epsilon/2}(f(x))$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ there exists a point $y \in (x-\delta, x) \cap \mathbb{Q}$ and a point $z \in (x, x+\delta) \cap \mathbb{Q}$. Since $y, z \in \mathbb{Q}$ we know that $\lim_{k\to\infty} f_{n_k}(y) = f(y)$ and $\lim_{k\to\infty} f_{n_k}(z) = f(z)$. So there exists $K \in \mathbb{N}$ such that for $k > K$ we have both $|f_{n_k}(y) - f(y)| \leq \epsilon/2$ and $|f_{n_k}(z) - f(z)| \leq \epsilon/2$. By the triangle inequality, for $k > K$ we have $|f_{n_k}(y) - f(x)| \leq \epsilon$ and $|f_{n_k}(z) - f(x)| \leq \epsilon$. Since f_{n_k} is an increasing function, $f_{n_k}(y) < f_{n_k}(x) < f_{n_k}(z)$. Combining these results we have for $k > K$

$$f_{n_k}(x) - f(x) \leq f_{n_k}(z) - f(x) \leq \epsilon \qquad \text{and} \qquad f(x) - f_{n_k}(x) \leq f(x) - f_{n_k}(y) \leq \epsilon.$$

Therefore $|f_{n_k}(x) - f(x)| \leq \epsilon$ for all $k > K$. Since ϵ was arbitrary, this implies that $\lim_{k\to\infty} f_{n_k}(x) = f(x)$.

Since $f(x)$ is a monotone non-decreasing function on $\mathbb{R}$ we know by Theorem 4.30 that $f(x)$ has at most countably many discontinuities. Then again applying Theorem 7.23 we know that there exists a subsequence $\{f_{n_{k_i}}\}$ of $\{f_{n_k}\}$ such that $\{f_{n_{k_i}}(x)\}$ converges point-wise at every point x where f has a discontinuity. Define $h(x)$ to be equal to $f(x)$ where f is continuous and equal to the limit of $\{f_{n_{k_i}}(x)\}$ where f is discontinuous. Then $\{f_{n_{k_i}}\}$ converges point-wise to $h(x)$, as desired.

(b) We now assume that f is continuous. We will show that f_{n_k} converges uniformly to f on any closed and bounded interval $[a.b]$. Since any compact set $K \subseteq \mathbb{R}$ is contained in such an interval, this shows uniform convergence on all compact sets.

Fix any $\epsilon > 0$. Since $[a, b]$ is compact we know that f is uniformly continuous on $[a, b]$. So there exists $\delta > 0$ such that for $x, y \in [a, b]$ when $|x - y| < \delta$, $|f(x) - f(y)| < \epsilon/2$. Define $x_i = a + i \cdot \frac{\delta}{2}$ for $i = \{1, 2, \ldots, m-1\}$ where $m - 1$ is the largest natural number such that $x_i < b$. Let $x_m = b$. For each x_i there exists a N_i such that for $k > N_i$ we have $|f_{n_k}(x_i) - f(x_i)| \leq \epsilon/2$. Let $N = \max\{N_i\}_{i=1}^m$.

Now fix any point $y \in K$. Then there exists $j \in \{1, 2, \ldots, m-1\}$ such that $x_j \leq y \leq x_{j+1}$. So $|x_j - y| < \delta$ and $|x_{j+1} - y| < \delta$. Using the monotonicity of the f_n, for $k > N$ we have

$$f_{n_k}(y) - f(y) \leq f_{n_k}(x_{j+1}) - f(y) < f(x_{j+1}) - f(y) + \epsilon/2 < \epsilon$$

and

$$f(y) - f_{n_k}(y) \leq f(y) - f_{n_k}(x_j) < f(y) - f(x_j) + \epsilon/2 < \epsilon$$

Therefore $|f(y) - f_{n_k}(y)| < \epsilon$, so f_{n_k} converges uniformly to f on $[a, b]$.

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Exercise 13.1.

Note that for each $n \in \mathbb{N}$ and for all $x \in \mathbb{R}$ we have $|a_n \cos(nx)| = |a_n||\cos(nx)| \leq |a_n|$. Since $\sum_{n=1}^{\infty} |a_n|$ is convergent, by Weierstrass' criteria we know $\sum_{n=1}^{\infty} a_n cos(nx)$ converges uniformly for all $x \in \mathbb{R}$.

Exercise 13.2.

Fix any $x \in \mathbb{R}$. We will show that $\sum_{n=1}^{\infty} \left(\frac{x^n}{n!}\right)^2$ converges uniformly on $[x-1, x+1]$. Since for any finite $m \in N$, $\sum_{n=1}^{m} \left(\frac{x^n}{n!}\right)^2$ is continuous we will have that the infinite sum is a uniform limit of continuous functions and hence is continuous on $[x-1, x+1]$. Since x is arbitrary, this will show that the infinite sum is continuous on all of $\mathbb{R}$.

Let $M = \max\{|x-1|, |x+1|\}$. Then for any $y \in [x-1, x+1]$ we have $\left|\left(\frac{y^n}{n!}\right)^2\right| \leq \left(\frac{M^n}{n!}\right)^2$. So by Weierstrass' criteria, to show uniform convergence of $\sum_{n=1}^{\infty} \left(\frac{x^n}{n!}\right)^2$ on $[x-1, x+1]$ it is sufficient to show the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{M^n}{n!}\right)^2$. but the ratio of consecutive terms of this series is $\left(\frac{M^{n+1}}{(n+1)!}\right)^2 / \left(\frac{M^n}{n!}\right)^2 = \frac{M^2}{n^2}$, which approaches zero as $n \to \infty$. Therefore by the ratio test (Theorem 3.34) we know that $\sum_{n=1}^{\infty} \left(\frac{x^n}{n!}\right)^2$ converges.

Exercise 13.3.

For $m \in \mathbb{N}$, let $F_m = \sum_{n=1}^{m} f_n$ be the partial sums of the f_n. Since the F_m are uniformly bounded there exists $M > 0$ such that $|F_m(x)| < M$ for all $m \in \mathbb{N}$ and all $x \in E$. Fix any $\epsilon > 0$. Since $g_n \to 0$ uniformly on E, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in E$, $|g_n(x)| < \epsilon/2M$. Further note that for each $x \in E$, since $g_n(x)$ is decreasing and converges to zero, for each $n \in \mathbb{N}$, $g_n(x) \geq 0$. Then for $N \leq p \leq q$ and $x \in E$ we have

$$
\begin{aligned}
\left|\sum_{n=1}^{q}(f_n g_n)(x) - \sum_{n=1}^{p-1}(f_n g_n)(x)\right| &= \left|\sum_{n=p}^{q}(f_n g_n)(x)\right| \\
&= \left|\sum_{n=p}^{q-1} F_n(x)(g_n(x) - g_{n+1}(x)) + F_q(x)\cdot g_q(x) - F_{p-1}(x)\cdot g_p(x)\right| \\
&\leq M\left|\sum_{n=p}^{q-1}(g_n(x) - g_{n+1}(x)) + g_q(x) + g_p(x)\right| \\
&= M\left|g_p(x) - g_q(x) + g_q(x) + g_p(x)\right| \\
&= 2M g_p(x) < 2M\epsilon/2M = \epsilon
\end{aligned}
$$

In the second line we have applied Theorem 3.41. The inequality in the third line uses that fact that $g_n(x) - g_{n+1}(x) \geq 0$ for all $n \in \mathbb{N}$.

This shows that the partial sums of $\sum f_n g_n$ satisfy theorem 7.8, so $\sum f_n g_n$ converges uniformly on E.

Exercise 13.4.

For all $x \in \mathbb{R}$ and $n \in \mathbb{N}$ we have $\left|\frac{x^2}{(1+nx^2)\sqrt{n}}\right| \leq \frac{x^2}{nx^2\sqrt{n}} = \frac{1}{n^{3/2}}$.

By theorem 3.28 we know that $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges.

So applying Weierstrass' criteria we know that $\sum_{n=1}^{\infty} \frac{x^2}{(1+nx^2)\sqrt{n}}$ is uniformly convergent on all of $\mathbb{R}$. Therefore $\sum_{n=1}^{\infty} \frac{x^2}{(1+nx^2)\sqrt{n}}$ is uniformly convergent on $[-h, h]$ for all $h > 0$.

Exercise 13.5.

First fix any interval $[a, b] \subseteq [1, 2]$ where $a < b$. Since the irrationals are dense in $\mathbb{R}$ we have a sequence of irrationals in $[a, b]$ converging to b. Therefore $\sup_{x\in[a,b]} f(x) = 5b + 3$.
Now let P_n be the partition of $[1, 2]$ into n equal subintervals. Then we have

$$\begin{aligned} U(P_n, f) = \sum_{i=1}^{n} \frac{1}{n} \cdot \sup_{x\in[1+(i-1)/n,1+i/n]} f(x) &= \frac{1}{n} \sum_{i=1}^{n} (5(1 + i/n) + 3) \\ &= \frac{1}{n}\left(\sum_{i=1}^{n} 8 + \frac{5}{n}\sum_{i=1}^{n} i\right) \\ &= \frac{1}{n}\left(8n + \frac{5}{n} \cdot \frac{n(n+1)}{2}\right) \\ &= 8 + \frac{5n+5}{2n} = \frac{21}{2} + \frac{5}{2n} \end{aligned}$$

We know $\overline{\int_1^2} f(x)dx = \inf U(P, f)$ where the infimum is taken over all partitions of $[1, 2]$. Restricting to just the partitions P_n we have

$$\overline{\int_1^2} f(x)dx \leq \inf_{n\in\mathbb{N}} U(P_n, f) = \inf_{n\in\mathbb{N}} \left(\frac{21}{2} + \frac{5}{2n}\right) = \frac{21}{2}$$

Therefore $\frac{21}{2}$ is an upper bound for $\overline{\int_1^2} f(x)dx$.

Exercise 13.6.

Fix any interval $[a, b]$ where $a < b$. Then consider any c, d such that $a \leq c < d \leq b$. Since the rationals are dense in $\mathbb{R}$ there is some rational in $[c, d]$, so $\sup_{x\in[c,d]} f(x) = 1$. Since the irrationals are dense in $\mathbb{R}$ there is some irrational in $[c, d]$, so $\inf_{x\in[c,d]} f(x) = 0$. Then using the notation of section 6.1, if P is any partition of $[a, b]$ then $U(P, f) = \sum_{i=1}^{n} M_i\Delta x_i = \sum_{i=1}^{n} \Delta x_i = (b - a) > 0$. However, $L(P, f) = \sum_{i=1}^{n} m_i\Delta x_i = \sum_{i=1}^{n} 0 = 0$. Therefore $\overline{\int_a^b} f(x)dx = (b - a) \neq 0 = \underline{\int_a^b} f(x)dx$. So f is not Riemann integrable on $[a, b]$.

Exercise 13.7.

$U(P, f, \alpha) = \frac{-3}{4} \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} + \frac{3}{2} \cdot 1 + 3 \cdot 1 = \frac{33}{8}$
$L(P, f, \alpha) = \frac{-3}{2} \cdot \frac{1}{2} + \frac{-3}{4} \cdot \frac{1}{2} + 0 \cdot 1 + \frac{3}{2} \cdot 1 = \frac{3}{8}$

Exercise 13.8.

Fix any partition P of $[a, b]$. Since f is zero at all but one point of $[a, b]$, the infimum of f on any subinterval in the partition with non-zero length is zero. Therefore $L(P, f, \alpha) = 0$. Since this holds for all partitions of $[a, b]$, $\underline{\int_a^b} f d\alpha = 0$.
Fix any $\epsilon > 0$. Since α is continuous at x_0 there exists $\delta > 0$ such that for all $x \in (x_0 - \delta, x_0 + \delta) \cap [a, b]$, $|\alpha(x) - \alpha(x_0)| < \epsilon/2$. Let $y \in (x_0 - \delta, x_0) \cap [a, b]$ and $z \in (x_0, x_0 + \delta) \cap [a, b]$. (If $x = a$, let $y = a$ and if $x = b$

let $z = b$). Define the partition $P_\epsilon = \{a, x, y, b\}$. Then $U(P_\epsilon, f, \alpha) = \alpha(y) - \alpha(x) < \epsilon$. Since this holds for every $\epsilon > 0$ we have $\overline{\int_a^b} f d\alpha = \inf U(f, P, \alpha) \le 0$.
By Theorem 6.5, $\overline{\int_a^b} f d\alpha \ge \underline{\int_a^b} f d\alpha$. Therefore $\overline{\int_a^b} f d\alpha = \underline{\int_a^b} f d\alpha = 0$, so $f \in \mathscr{R}(\alpha)$.

Exercise 13.9.

(a) Let $m = \inf_{x \in I} f(x)$. Then for all $x \in I$, $m \le f(x)$. Therefore $k \cdot m \ge k \cdot f(x) = g(x)$ for all $x \in I$. So $k \cdot m$ is an upper bound for $\{g(x)\}_{x \in I}$.

Now consider any $c < k \cdot m$. Then $c/k > m$. Since m is the greatest lower bound of $\{f(x)\}_{x \in I}$, c/k is not lower bound for $\{f(x)\}_{x \in I}$, so there is some $y \in I$ such that $f(y) < c/k$. Then $c < k \cdot f(y) = g(y)$ So c is not an upper bound for $\{g(x)\}_{x \in I}$.

Therefore $k \cdot m = \sup_{x \in I} g(x)$.

Note that this argument can also be used to prove the following more general result:

Let $S \subseteq \mathbb{R}$ be bounded and let k be a fixed negative number. Then $\sup\{kx : x \in S\} = k \cdot \inf\{x : x \in S\}$ and $\inf\{kx : x \in S\} = k \cdot \sup\{x : x \in S\}$.

(b) Fix any partition P of I. By the result of part (a) we know that for each subinterval $I_j := [x_{j-1}, x_j]$ of P, $\sup_{x \in I_j} g(x) = k \cdot \inf_{x \in I_j} f(x)$. Therefore

$$U(P, g) = \sum_{i=1}^{n} \sup_{x \in I_j} g(x) \Delta x_i = \sum_{i=1}^{n} k \cdot \inf_{x \in I_j} f(x) \Delta x_i = k \cdot \sum_{i=1}^{n} \inf_{x \in I_j} f(x) \Delta x_i = k \cdot L(P, f)$$

Since this applies to all partitions P of I we have

$$U(g) = \inf U(P, g) = \inf(k \cdot L(P, f)) = k \cdot \sup(L(P, f)) = k \cdot L(f)$$

In the third equality we have used the generalized result from part (a).

By similar argument with the supremum and infimum interchanged we also have that $L(g) = k \cdot U(f)$.

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Exercise 14.1.

This is exercise 13.9

Exercise 14.2.

Let P be any partition of $[a,b]$. Then since we are integrating the constant function $f(x)=1$, the supremum and infimum over each subinterval of the partition is 1. Therefore $U(P,f,\alpha)=\sum_{i=1}^{n}(\alpha(x_i)-\alpha(x_{i-1}))$. This is a telescoping sum which yields $U(P,f,\alpha)=\alpha(x_n)-\alpha(x_0)=\alpha(b)-\alpha(a)$. Similarly, $L(P,f,\alpha)=\alpha(b)-\alpha(a)$. Since this holds for all partitions P of $[a,b]$ we have

$$\overline{\int_a^b} d\alpha = \inf U(P,f,\alpha) = \alpha(b)-\alpha(a) = \sup L(P,f,\alpha) = \underline{\int_a^b} d\alpha$$

Therefore the constant function 1 is in $\mathscr{R}(\alpha)$ and $\int_a^b d\alpha = \alpha(b)-\alpha(a)$.

Exercise 14.3.

Let $f:[a,b]\to[0,\infty)$ be a continuous function such that $\int_a^b f(x)dx = 0$. Assume that there exists a point $p\in[a,b]$ such that $f(p)\neq 0$. Then by continuity, there exists $\delta>0$ such that for all $y\in N_\delta(p)$, $|f(y)-f(p)|<|f(p)|/2$. So for $y\in N_\delta(p)$, $|f(y)|>|f(p)|/2$.
If $p\notin\{a,b\}$, let $\delta'=\min\{\delta, b-p, p-a\}$. Then

$$\int_a^b f(x)dx = \int_a^b |f(x)|dx \geq \int_{p-\delta'}^{p+\delta'} |f(x)|dx \geq \frac{|f(P)|}{2}\cdot 2\delta' > 0, \tag{2}$$

contradicting the assumption that $\int_a^b f(x)=0$. Therefore f must be exactly zero on $[a,b]$.
If p is either a or b then the calculation (2) can be repeated with $\delta'=\min\{\delta,b-a\}$ and with the δ' neighborhood about p now extending in only one direction from p.

Exercise 14.4.

(a) Given an interval $[a,b]$ and a finite collection of points $\{y_i\}_{i=1}^N\in[a,b]$, define $f:[a,b]\to\mathbb{R}$ by $f(x)=1$ for all $x\in\{y_i\}_{i=1}^N$ and $f(x)=0$ otherwise. Let P_n be the partition of $[a,b]$ into n equal intervals for $n\in\mathbb{N}$. There are only N points where f has non-zero value and at all these points f has value 1. Each of these points can be in at most 2 of the intervals of our partition, so

$$U(P_n,f)\leq\frac{b-a}{n}\cdot 2N \qquad \text{and} \qquad L(P_n,f)\geq 0. \tag{3}$$

Therefore $U(P_n,f)-L(P_n,f)\leq\frac{b-a}{n}\cdot 2N$. So given any $\epsilon>0$, taking $n\geq\frac{(b-a)2N}{\epsilon}$ we have $U(P_n,f)-L(P_n,f)\leq\epsilon$. Therefore by Theorem 6.6 f is integrable on $[a,b]$. From (3) we see that $\inf U(P,f)\leq 0$ and $\sup L(P,f)\geq 0$ where the infimum and supremum are taken over all partitions P of $[a,b]$. Since f is Riemann integrable these two values must both be equal to $\int_a^b f(x)dx$. Hence $\int_a^b f(x)dx = 0$.

(b) Since $\mathbb{Q} \cap [0,1]$ is countable we can list its elements. Let $\{x_i\}_{i=1}^{\infty} = \mathbb{Q} \cap [0,1]$ where all the x_i are distinct. Define $f_n(x) = 1$ for all $x \in \{x_i\}_{i=1}^{n}$ and $f_n(x) = 0$ otherwise. Then for all $n \in \mathbb{N}$, $f_n(x) = 0$ for all irrational x. For any $x_i \in \mathbb{Q} \cap [0,1]$, $f_n(x_i) = 1$ for all $n \geq i$. Therefore the $\{f_n\}_{n=1}^{\infty}$ converge point-wise to the g.

From exercise 13.6 we know that $g(x)$ is not Riemann integrable, therefore $\int_0^1 \lim_{n\to\infty} f_n(x)dx$ does not exist. However, by part (a), for all $n \in \mathbb{N}$ we know that $\int_0^1 f_n(x)dx = 0$, so $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$.

Later in the course we will see that $f(x)$ is Lebesgue integrable, and using Lebesgue integration

$$\int_0^1 \lim_{n\to\infty} f_n(x)dx = \int_0^1 f(x)dx = 0 = \lim_{n\to\infty} \int_0^1 f_n(x)dx.$$

This ability to interchange limits and Lebesgue integrals is an application of Lebesgue's dominated convergence theorem. This is one of the ways in which Lebesgue integration is 'nicer' than Riemann integration.

Exercise 14.5.

Let f and g be Riemann integrable on $[a,b]$. Since f, g are Riemann integrable they are both bounded on $[a,b]$. Fix M such that $|f(x)| < M$ and $|g(x)| < M$ for all $x \in [a,b]$. Fix $\epsilon > 0$. Then by taking refinements we know that there exists a partition $P = \{x_1, x_2, \ldots, x_n\}$ of $[a,b]$ such that $U(P,f) - L(P,f) \leq \epsilon/4M$ and $U(P,g) - L(P,g) \leq \epsilon/4M$. For any function $h : [a,b] \to \mathbb{R}$, let $M_i^h = \sup_{x\in[x_i,x_{i+1}]} h(x)$ and $m_i^h = \inf_{x\in[x_i,x_{i+1}]} h(x)$. Then for $i = 1, 2, \ldots, n-1$, there exists points $y_i, z_i \in [a,b]$ such that $M_i^{fg} - f(y_i)g(y_i) < \frac{\epsilon}{4(b-a)}$ and $f(z_i)g(z_i) - m_i^{fg} < \frac{\epsilon}{4(b-a)}$.
Then we have

$$\begin{aligned}
U(P,fg) - L(P,fg) &= \sum_{i=1}^{n-1} \left(M_i^{fg} - m_i^{fg}\right)(x_{i+1} - x_i) \\
&< \sum_{i=1}^{n-1} \left(f(y_i)g(y_i) - f(z_i)g(z_i)\right)(x_{i+1} - x_i) + 2(b-a) \cdot \frac{\epsilon}{4(b-a)} \\
&= \sum_{i=1}^{n-1} \left(f(y_i)g(y_i) - f(z_i)g(y_i) + f(z_i)g(y_i) - f(z_i)g(z_i)\right)(x_{i+1} - x_i) + \frac{\epsilon}{2}
\end{aligned}$$

Breaking the sum into two pieces and factoring yields

$$\begin{aligned}
U(P,fg) - L(P,fg) &< \sum_{i=1}^{n-1} g(y_i)\left(f(y_i) - f(z_i)\right)(x_{i+1} - x_i) + \sum_{i=1}^{n-1} f(z_i)\left(g(y_i) - g(z_i)\right)(x_{i+1} - x_i) + \frac{\epsilon}{2} \\
&\leq M \sum_{i=1}^{n-1} \left(M_i^f - m_i^f\right)(x_{i+1} - x_i) + M \sum_{i=1}^{n-1} \left(M_i^g - m_i^g\right)(x_{i+1} - x_i) + \frac{\epsilon}{2} \\
&= M(U(P,f) - L(P,f)) - M(U(P,g) - L(P,g)) + \frac{\epsilon}{2} \\
&\leq 2M \cdot \frac{\epsilon}{4M} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Therefore fg is Riemann integrable on $[a, b]$.
An alternative approach is to prove that for any Riemann integrable function f, f^2 is also Riemann integrable, and then to use this result to consider the product of two Riemann integrable functions.

Lemma. *Let f be a Riemann integrable function on $[a, b]$. Then f^2 is also Riemann integrable on $[a, b]$.*

Proof. Fix any $\epsilon > 0$. Since f is Riemann integrable there exists B such that $|f(x)| < B$ for all $x \in [a, b]$. Also, f Riemann integrable implies that $|f|$ is Riemann integrable. So there exists a partition $P = \{x_1, x_2, \ldots, x_n\}$ of $[a, b]$ such that $U(P, |f|) - L(P, |f|) \leq \epsilon/(2B)$. Then

$$\begin{aligned} U(P, f^2) - L(P, f^2) &= \sum_{i=1}^{n-1} \left(M_i^{f^2} - m_i^{f^2}\right)(x_{i+1} - x_i) \\ &= \sum_{i=1}^{n-1} \left(M_i^{|f|^2} - m_i^{|f|^2}\right)(x_{i+1} - x_i) \\ &= \sum_{i=1}^{n-1} \left(\left(M_i^{|f|}\right)^2 - \left(m_i^{|f|}\right)^2\right)(x_{i+1} - x_i) \end{aligned}$$

Breaking this up as the difference of two squares yields

$$\begin{aligned} U(P, f^2) - L(P, f^2) &= \sum_{i=1}^{n-1} \left(M_i^{|f|} + m_i^{|f|}\right)\left(M_i^{|f|} - m_i^{|f|}\right)(x_{i+1} - x_i) \\ &\leq \sum_{i=1}^{n-1} 2B \left(M_i^{|f|} - m_i^{|f|}\right)(x_{i+1} - x_i) \\ &= 2B\left(U(P, |f|) - L(P, |f|)\right) \\ &\leq \epsilon \end{aligned}$$

Therefore, by Theorem 6.6 we know that f^2 is Riemann integrable. ☐

Note that this lemma could also be proved using Rudin theorem 6.11.
Now let f, g be two Riemann integrable functions on $[a, b]$. Then $4fg = (f + g)^2 - (f - g)^2$. We have already shown that sums and differences of Riemann integrable functions are Riemann integrable. Combining this with the above lemma we see that $(f + g)^2 - (f - g)^2$ is Riemann integrable on $[a, b]$. Then since scalar multiples of Riemann integrable functions are Riemann integrable we have that fg is Riemann integrable on $[a, b]$.

Exercise 14.6.
Since f is continuous on $[a.b]$ it is also integrable on $[a, b]$. Let $k = \int_a^b f(x)dx$. Then $f(x) - k$ is a continuous function on $[a, b]$ with $\int_a^b (f(x) - k)dx = 0$, so we have

$$\int_a^b (f(x) - k)^2 dx = \int_a^b f(x)(f(x) - k)dx - k\int_a^b (f(x) - k)dx = 0 - k \cdot 0 = 0$$

However, $(f(x) - k)^2 \geq 0$ for all $x \in [a, b]$. So by Exercise 14.3 we know that $f(x) - k = 0$ for all $x \in [a, b]$. Therefore $f(x) = k$ for all $x \in [a, b]$, so f is a constant function.

Exercise 14.7.

Since f is non-negative, $L(P,f)$ is non-negative for every partition P of $[0,1]$. So by Theorem 6.6, if for every $\epsilon > 0$ we can find a partition P_ϵ such that $U(P,f) < \epsilon$, then $f \in \mathscr{R}$ and $\int_a^b f(x)dx = 0$.

So fix any $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $1/N < \epsilon/2$. For $n \in \mathbb{N}$, let $A_n = \{x \in \mathbb{Q} \cap (0,1] : x = m/n, m \in \mathbb{N}\}$. Then $|A_n| = n$. Let $B_n = \cup_{i=1}^n A_i$. Then B_n is finite for all $n \in \mathbb{N}$. Let $k = |B_N|$. Then for $x \in [0,1] \backslash B_N$, $f(x) < \epsilon/2$.

Fix $m \in \mathbb{N}$ such that $m > 4k/\epsilon$. Let P_m be the partition of $[a,b]$ into m equal intervals. There are only k points where f has value greater than $\epsilon/2$ and at these points f has value less than or equal to 1. Each of these points can be in at most 2 of the intervals of our partition, so

$$U(P_m, f) \le \frac{1}{m} \cdot 2k + \frac{1}{m} \cdot \frac{\epsilon}{2} \cdot m = \frac{2k}{m} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

As argued above, this implies the desired result.

Exercise 14.8.

(a) For any $c \in (0,1)$, $\int_0^1 f(x)dx = \int_0^c f(x)dx + \int_c^1 f(x)dx$. Therefore, it is sufficient to show that $\lim_{c \to 0^+} \int_0^c f(x)dx = 0$.

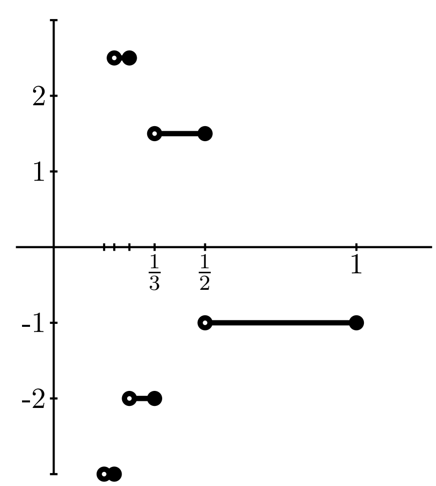
Since f is Riemann integrable on $[0,1]$, f is bounded on $[0,1]$. So there exists an M such that $|f(x)| < M$ for all $x \in [0,1]$. Then

$$\lim_{c \to 0^+} \left| \int_0^c f(x)dx \right| \le \lim_{c \to 0^+} \int_0^c |f(x)|dx \le \lim_{c \to 0^+} \int_0^c M dx = \lim_{c \to 0^+} cM = 0.$$

Therefore $\lim_{c \to 0^+} \int_0^c f(x) = 0$, as desired.

(b) As motivation for this problem, note that the series $\sum \frac{(-1)^n}{n}$ converges (by the alternating series test) while the series $\sum \frac{1}{n}$ diverges (Theorem 3.28). Define $f : (0,1] \to \mathbb{R}$ by $f(x) = (-1)^n \cdot (n+1)$ for $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$ for all $n \in \mathbb{N}$. Then by Exercise 14.4 we can ignore the value of f at $\frac{1}{n+1}$ and derive

$$\int_{\frac{1}{n+1}}^{\frac{1}{n}} f(x)dx = \left(\frac{1}{n} - \frac{1}{n+1}\right)(-1)^n \cdot (n+1) = \frac{1}{n(n+1)}(-1)^n \cdot (n+1) = \frac{(-1)^n}{n}. \quad (4)$$



Fix any $\epsilon > 0$. Let $L = \sum_{n=1}^\infty \frac{(-1)^n}{n}$. Then there exists an $M \in \mathbb{N}$ such that for $m \ge M$, $\left| L - \sum_{n=1}^m \frac{(-1)^n}{n} \right| \le \epsilon/2$. Fix $M' \in \mathbb{N}$ such that $M' \ge M$ and $1/M' \le \epsilon/2$. Fix any $c \in \left(0, \frac{1}{M'+1}\right]$.

Then there exists a $m \in \mathbb{N}$ such that $c \in \left(\frac{1}{m+2}, \frac{1}{m+1}\right]$. So $m \geq M' \geq M$ and $1/(m+1) \leq \epsilon/2$. Then by (4)

$$\int_c^1 f(x)dx = \int_c^{\frac{1}{m+1}} f(x)dx + \sum_{n=1}^{m} \int_{\frac{1}{n+1}}^{\frac{1}{n}} f(x)dx = \left(\frac{1}{m+1} - c\right) \cdot (-1)^{m+1} \cdot (m+2) + \sum_{n=1}^{m} \frac{(-1)^n}{n}.$$

Therefore

$$\begin{aligned}
\left|L - \int_c^1 f(x)dx\right| &\leq \left|L - \sum_{n=1}^{m} \frac{(-1)^n}{n}\right| + \left|\left(\frac{1}{m+1} - c\right) \cdot (-1)^{m+1} \cdot (m+2)\right| \\
&\leq \frac{\epsilon}{2} + \left|\left(\frac{1}{m+1} - \frac{1}{m+2}\right) \cdot (m+2)\right| \\
&= \frac{\epsilon}{2} + \left|\left(\frac{1}{(m+1)(m+2)}\right) \cdot (m+2)\right| \\
&= \frac{\epsilon}{2} + \left|\frac{1}{m+1}\right| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Therefore $\lim_{c \to 0} \int_c^1 f(x)dx = L$.

However, for $m \in \mathbb{N}$ by (4) we have

$$\int_{\frac{1}{m+1}}^1 |f(x)|dx = \sum_{n=1}^{m} \int_{\frac{1}{n+1}}^{\frac{1}{n}} |f(x)|dx = \sum_{n=1}^{m} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{m} \frac{1}{n}$$

Since $\sum \frac{1}{n}$ diverges, $\lim_{m \to \infty} \int_{\frac{1}{m+1}}^1 |f(x)|dx$ diverges.

Therefore by Theorem 4.2 we have that $\lim_{c \to 0} \int_c^1 |f(x)|dx$ does not exist.

The function $g(x) = \frac{1}{x} \sin\left(\frac{1}{x}\right)$ provides an example of a continuous function with the desired properties.

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Exercise 15.1.

Since $G(x)$ is an anti-derivative of f we know $G'(x) = f(x)$ for all $x \in [a, b]$. So by the argument of Theorem 7.10 for F we know that $G(b) - G(a) = \int_a^b f(x)dx$.

Exercise 15.2.

Since g is continuous on $[a, b]$ it obtains its maximum M and minimum m on this interval.

Then since $f \geq 0$ we have

$$m \int_a^b f(x)dx = \int_a^b m \cdot f(x)dx \leq \int_a^b f(x)g(x)dx \leq M \cdot \int_a^b f(x)Mdx = M \int_a^b f(x)dx \tag{5}$$

If $\int_a^b f(x)dx = 0$ then by (5) we have $\int_a^b f(x)g(x)dx = 0$, so the desired equality holds for all $c \in [a.b]$. If $\int_a^b f(x)dx \neq 0$ then since $f \geq 0$ we must have $\int_a^b f(x)dx > 0$, so (5) can be rewritten as

$$m \leq \frac{\int_a^b f(x)g(x)dx}{\int_a^b f(x)dx} \leq M$$

Then since g is continuous, by the intermediate value theorem, there exists $c \in [a, b]$ such that

$$g(c) = \frac{\int_a^b f(x)g(x)dx}{\int_a^b f(x)dx} \tag{6}$$

Rearranging (6) yields the desired equality.

Exercise 15.3.

(a) This result follow from the more general argument of part (b) with $g(x) = x^2$. Using the result of part (b) followed by the Fundamental Theorem of Calculus we have

$$F'(x) = \frac{d}{dx} \int_0^{x^2} f(s)ds = \frac{d}{dx} \int_0^x 2xF(x^2)dx = 2xF(x^2)$$

(b) Let $g : [a, b] \to [c, d]$ be a differentiable function such that g' is Riemann integrable on $[a, b]$. Let f be a continuous function on $[c, d]$. Define $F(x) = \int_c^x f(t)dt$ for $x \in [c, d]$. Then by Theorem 6.20 we know that F is differentiable on $[c, d]$ with $F'(x) = f(x)$. By the chain rule (Theorem 5.5), $F \circ g$ is differentiable on $[a, b]$ with $(F \circ g)'(x) = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x)$. Since f, g are continuous, $f \circ g$ is continuous on $[a, b]$ and hence Riemann integrable. Since g' is also Riemann integrable on $[a, b]$, exercise 14.5 shows that $f(g(x)) \cdot g'(x)$ is Riemann integrable on $[a, b]$. Then by the fundamental theorem of calculus (Theorem 6.21) we have

$$\int_a^b f(g(x)) \cdot g'(x)dx = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(x)dx,$$

which is the desired result.

Exercise 15.4.

(a) Since $\|u\|_2$ is non-negative for any Riemann integrable function u, it is sufficient to verify the square of the desired inequality. From the triangle inequality we have

$$\|f+g\|_2^2 = \int_a^b |f+g|^2 dx \leq \int_a^b \left(|f|^2 + 2|fg| + |g|^2 dx\right) dx = \|f\|_2^2 + 2\int_a^b |fg| dx + \|g\|_2^2.$$

Combining this with the Schwarz inequality yields

$$\|f+g\|_2^2 \leq \|f\|_2^2 + 2\|f\|_2 \cdot \|g\|_2 + \|g\|_2^2 = (\|f\|_2 + \|g\|_2)^2.$$

As noted above, this implies the desired result.

(b) Now let $f, g, h \in \mathcal{R}$. Then by part (a) we have

$$\|f-h\|_2 = \|(f-g)+(g-h)\|_2 \leq \|f-g\|_2 + \|g-h\|_2,$$

which is the desired inequality.

This shows that $\|f-g\|_2$ obeys the triangle inequality. However, this is not a metric on the set of Riemann integrable functions because there are non-zero Riemann integrable functions that have integral zero (see exercise 14.4). Since $\|f-g\|_2$ is non-negative, symmetric, and obeys the triangle inequality it is called a *pseudo-metric* on $\mathscr{R}$. Using the ideas from Rudin problem 6.2 one can show that $\|f-g\|_2$ does give a metric on the set of continuous functions on $[a,b]$. When we study Lebesgue theory we will define $\|f-g\|_2$ as a metric on equivalence classes of Lebesgue square integrable functions, where two functions are equivalent if they disagree only on a set of measure zero.

Exercise 15.5.

Let $f \in \mathscr{R}$. Fix $\epsilon > 0$. Let M be a bound for $|f|$ on $[a,b]$. Then we know that there exists a partition $P = \{x_1, x_2, \ldots, x_n\}$ of $[a,b]$ such that $U(P,f) - L(P,f) < \epsilon^2/(2M)$. Define $g : [a,b] \to \mathbb{R}$ by

$$g(t) = \frac{x_{i+1}-t}{x_{i+1}-x_i} f(x_i) + \frac{t-x_i}{x_{i+1}-x_i} f(x_{i+1}) \qquad t \in [x_i, x_{i+1}].$$

Then g is a piece-wise linear and continuous function. On each interval $[x_i, x_{i+1}]$, the value of $g(t)$ is always between $f(x_i)$ and $f(x_{i+1})$. So in particular, for all $t \in [x_i, x_{i+1}]$, $|f(t) - g(t)| \leq (M_i^f - m_i^f) \leq 2M$. Therefore $0 \leq L(P, |f-g|) \leq U(P, |f-g|) \leq U(P,f) - L(P,f) < \epsilon^2/(2M)$. So $\int_a^b |f-g| dx < \epsilon^2/(2M)$. Since $|f-g| < 2M$ on $[a,b]$, $\int_a^b |f-g|^2 dx \leq 2M \int_a^b |f-g| dx < 2M \cdot \epsilon^2/(2M) = \epsilon^2$. Therefore $\|f-g\|_2 < \epsilon$, as desired.

This shows that the continuous functions are 'dense' in the set of Riemann integrable functions using the pseudo-metric $\|f-g\|_2$.

Exercise 15.6.

(a) Assume that $|f'(x)| < M$ for all $x \in (a, b)$. Then since $f(a) = 0$, by the lemma of exercise 10.2, for $x \in [a, ((a + b)/2]$ we have $|f(x)| < M(x - a)$. Similarly, since $f(b) = 0$, for $x \in [(a + b)/2, b]$ we have $|f(x)| < M(b - x)$. Combining these results yields

$$\begin{aligned}
\int_a^b f(x)dx \leq \int_a^b |f(x)|dx &= \int_a^{(a+b)/2} |f(x)|dx + \int_{(a+b)/2}^b |f(x)|dx \\
&< \int_a^{(a+b)/2} M(x-a)dx + \int_{(a+b)/2}^b M(b-x)dx \\
&= M\left(\left[\frac{x^2}{2} - ax\right]_a^{(a+b)/2} + \left[bx - \frac{x^2}{2}\right]_{(a+b)/2}^b\right) \\
&= \frac{M}{4}\left(\frac{(a+b)^2}{2} - 2a(a+b) + 2a^2 + 2b^2 - 2b(a+b) + \frac{(a+b)^2}{2}\right) \\
&= \frac{M}{4}\left(a^2 - 2ab + b^2\right) \\
&= \frac{M(a-b)^2}{4}
\end{aligned}$$

So $\frac{4}{(a-b)^2}\int_a^b f(x)dx < M$. Therefore, there must be some $c \in (a, b)$ such that $|f'(c)| \geq \frac{4}{(a-b)^2}\int_a^b f(x)dx$.

(b) Equality is not possible in part (a) in the sense that there will always be some point $c \in (a, b)$ such that the inequality is strict.

Assume that $f(x)$ satisfies the conditions of part (a) with $\frac{4}{(b-a)^2}\int_a^b f(x)dx \geq |f'(x)|$ for all $x \in [a, b]$. Let $M = \frac{4}{(b-a)^2}\int_a^b f(x)dx$. Since f is non-constant, $f'(x)$ is not identically zero, so $M > 0$. Define

$$g(x) = \begin{cases} M(x-a) & x \in [a, \frac{a+b}{2}] \\ M(b-x) & x \in [\frac{a+b}{2}, b] \end{cases}$$

From the calculation of part (a) we know that $\int_a^b g(x)dx = \frac{M(a-b)^2}{4}$. By the lemma of exercise 10.2, $f(x) \leq g(x)$ for all $x \in [a, b]$.

Assume there exists a point $d \in [a, b]$ such that $f(d) \neq g(d)$. Then by exercise 14.3, since $g - f$ is continuous, non-negative, and not identically zero we have

$$\int_a^b g(x)dx - \int_a^b f(x)dx = \int_a^b (g(x) - f(x))dx > 0$$

Therefore

$$M = \frac{4}{(b-a)^2}\int_a^b f(x)dx < \frac{4}{(b-a)^2}\int_a^b g(x)dx = M$$

Contradiction. Therefore $f(x) = g(x)$ for all $x \in [a, b]$. But then $f(x)$ is not differentiable at $\frac{a+b}{2}$. Contradiction. Therefore no such f can exist and the inequality in part (a) is strict.

Exercise 15.7.

Define $F(x) = c + \int_a^x g(t)f(t)dt$. So $f(x) \leq F(x)$ for all $x \in [a,b]$. By the fundamental theorem of calculus, since f, g are continuous, $F'(x) = g(x)f(x)$. Since f, g, F are all non-negative we have

$$\frac{F'(x)}{F(x)} = \frac{g(x)f(x)}{F(x)} \leq \frac{g(x)f(x)}{f(x)} = g(x) \tag{7}$$

Integrating both sides of (7) from a to y where $y \in [a,b]$ we have

$$\begin{aligned}
\int_a^y \frac{F'(x)}{F(x)}dx &\leq \int_a^y g(x)dx \\
\ln F(y) - \ln F(a) &\leq \int_a^y g(x)dx \\
\ln F(y) - \ln c &\leq \int_a^y g(x)dx \\
\frac{F(y)}{c} &\leq e^{\int_a^y g(x)dx} \\
F(y) &\leq ce^{\int_a^y g(x)dx}
\end{aligned}$$

Since $f(y) \leq F(y)$ we have $f(y) \leq ce^{\int_a^y g(x)dx}$, as desired.

Exercise 15.8.

(a) Define $F(x) = \int_a^x f(x)dx$ for $x \in [a,b]$. Then

$$\lim_{n\to\infty} \int_{a_n}^b f(x)dx = \lim_{n\to\infty} \left(\int_a^b f(x)dx - \int_a^{a_n} f(x)dx \right) = \lim_{n\to\infty} (F(b) - F(a_n)) = F(b) - \lim_{n\to\infty} F(a_n)$$

By Theorem 6.20 we know that F is continuous on $[a.b]$. So from the sequential characterization of continuity we have

$$\lim_{n\to\infty} \int_{a_n}^b f(x)dx = F(b) - \lim_{n\to\infty} F(a_n) = F(b) - F(a) = F(b) - 0 = \int_a^b f(x)dx$$

(b) Since f is monotonic on $[0,1]$, f is integrable on $[a,b]$ (Theorem 6.9). Let $a_n = 2^{-(n+1)}$. Then by part (a) we know $\int_0^1 f(x)dx = \lim_{n\to\infty} \int_{a_n}^1 f(x)dx$.

For fixed $n \in \mathbb{N}$ we know that

$$\int_{a_n}^1 f(x)dx = \sum_{i=0}^n \int_{a_i}^{a_{i-1}} f(x)dx = \sum_{i=0}^n \int_{2^{-(i+1)}}^{2^{-i}} 2^{-i}dx = \sum_{i=0}^n 2^{-(i+1)} \cdot 2^{-i} = \frac{1}{2}\sum_{i=1}^n 4^{-i}$$

This is a geometric series, so by Theorem 3.26 we have

$$\int_0^1 f(x)dx = \lim_{n\to\infty} \frac{1}{2}\sum_{i=1}^n 4^{-i} = \frac{1}{2} \cdot \frac{1}{1 - 1/4} = \frac{2}{3}$$

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Exercise 16.1. This is exercise 14.6

Exercise 16.2. This is exercise 14.7

Exercise 16.3. This is exercise 14.8

Exercise 16.4.

(a) Using the fact that $\mathcal{M}_1$ and $\mathcal{M}_2$ are σ-algebras we will verify the three properties of a σ-algebra for $\mathcal{M}_1 \cap \mathcal{M}_2$.

  (i) Since $\mathbb{R} \in \mathcal{M}_1$ and $\mathbb{R} \in \mathcal{M}_2$, $\mathbb{R} \in \mathcal{M}_1 \cap \mathcal{M}_2$.

  (ii) Let $A \in \mathcal{M}_1 \cap \mathcal{M}_2$. Then $A \in \mathcal{M}_1$, so $A^c \in \mathcal{M}_1$. Also $A \in \mathcal{M}_2$, so $A^c \in \mathcal{M}_2$. Therefore $A^c \in \mathcal{M}_1 \cap \mathcal{M}_2$.

  (iii) Let $\{A_n\} \in \mathcal{M}_1 \cap \mathcal{M}_2$. Then $\{A_n\} \in \mathcal{M}_1$, so $\cup A_n \in \mathcal{M}_1$. Also $\{A_n\} \in \mathcal{M}_2$, so $\cup A_n \in \mathcal{M}_2$. Therefore $\cup A_n \in \mathcal{M}_1 \cap \mathcal{M}_2$.

Therefore $\mathcal{M}_1 \cap \mathcal{M}_2$ is a σ-algebra.

(b) Let $\{\sigma_\alpha\}_{\alpha \in I}$ be the set of all σ-algebras containing A, where I is some index set. Since the collection of all subsets of $\mathbb{R}$ is a σ-algebra containing A we know that $\{\sigma_\alpha\}_{\alpha \in I}$ is non-empty. Therefore, since each of the σ_α contains A, $A \subseteq \cap_{\alpha \in I} \sigma_\alpha$.

The arguments of part (a) can be applied without change to arbitrary intersections of σ-algebras. Therefore $\cap_{\alpha \in I} \sigma_\alpha$ is also a σ-algebra, as desired.

Exercise 16.5.
Since $\mathcal{M}$ is closed under complements and intersections, $F \backslash E = F \cap E^c \in \mathcal{M}$. Since $E \subseteq F$ we have $F = (F \backslash E) \cup E$. Also $(F \backslash E) \cap E = \emptyset$, so we know

$$\mu(F) = \mu((F \backslash E) \cup E) = \mu(F \backslash E) + \mu(E) \tag{8}$$

Since $\mu(E) < \infty$ we can subtract $\mu(E)$ from both sides of (8) which yields the desired result.
Note that since $\mu(F \backslash E) \geq 0$, (8) implies that $\mu(F) \geq \mu(E)$. This is still valid in the case where $\mu(E) = \infty$, in which case $\mu(F)$ is also ∞.

Exercise 16.6.

(a) Define $E_0 = \emptyset$. Fix any $i, j \in \mathbb{N}$, $i < j$. Then $(E_i \backslash E_{i-1}) \subseteq E_i \subseteq E_{j-1}$, but $(E_j \backslash E_{j-1}) \subseteq E_{j-1}^c$. Therefore $(E_i \backslash E_{i-1}) \cap (E_j \backslash E_{j-1}) = \emptyset$. So $\{E_i \backslash E_{i-1}\}_{i=1}^{\infty}$ are disjoint.

Additionally, we claim that $\cup_{i=1}^{\infty} E_i = \cup_{i=1}^{\infty} (E_i \backslash E_{i-1})$.

$\subseteq$ Let $x \in \cup_{i=1}^{\infty} E_i$. Then $x \in E_j$ for some $j \in \mathbb{N}$. So there exists some k that is the smallest element of $\mathbb{N}$ such that $x \in E_k$. Then $x \notin E_{k-1}$. So $x \in E_k \backslash E_{k-1}$. Therefore $x \in \cup_{i=1}^{\infty}(E_i \backslash E_{i-1})$.

$\supseteq$ Let $x \in \cup_{i=1}^{\infty}(E_i \backslash E_{i-1})$ Then $x \in E_j \backslash E_{j-1}$ for some $j \in \mathbb{N}$. Therefore $x \in E_j$, so $x \in \cup_{i=1}^{\infty} E_j$.

By similar argument $E_n = \cup_{i=1}^{n} E_i = \cup_{i=1}^{n}(E_i \backslash E_{i-1})$.

Using the properties of a measure and the above results we have

$$\mu\left(\cup_{i=1}^{\infty} E_i\right) = \mu\left(\cup_{i=1}^{\infty}(E_i \backslash E_{i-1})\right) = \sum_{i=1}^{\infty} \mu(E_i \backslash E_{i-1}) \tag{9}$$

$$\mu\left(E_n\right) = \mu\left(\cup_{i=1}^{n}(E_i \backslash E_{i-1})\right) = \sum_{i=1}^{n} \mu(E_i \backslash E_{i-1}) \tag{10}$$

Combining (9) and (10) with the definition of an infinite sum we have

$$\mu\left(\cup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i \backslash E_{i-1}) = \lim_{n \to \infty} \sum_{i=1}^{n} \mu(E_i \backslash E_{i-1}) = \lim_{n \to \infty} \mu(E_n)$$

(b) (i) Let $G_n = F_1 \backslash F_n$. Then $G_1 \subseteq G_2 \subseteq G_2 \subseteq \cdots$.

By the comment following exercise 16.5, since $\mu(F_1) < \infty$ we know that $\mu(F_n) \leq \mu(F_1) < \infty$ for all $n \in \mathbb{N}$ and similarly $\mu(\cap_{i=1}^{\infty} F_i) < \infty$. So applying the result of exercise 16.5, $\mu(G_n) = \mu(F_1) - \mu(F_n)$ for all $n \in \mathbb{N}$, and $\mu\left(F_1 \backslash \cap_{i=1}^{\infty} F_i\right) = \mu(F_1) - \mu\left(\cap_{i=1}^{\infty} F_i\right)$.

By DeMorgan's law we know $F_1 \backslash \left(\cap_{i=1}^{\infty} F_i\right) = \cup_{i=1}^{\infty}\left(F_1 \backslash F_i\right) = \cup_{i=1}^{\infty} G_i$.

Applying part (a) and the above results yields

$$\mu(F_1) - \mu\left(\cap_{i=1}^{\infty} F_i\right) = \mu(\cup_{i=1}^{\infty} G_i) = \lim_{n \to \infty} \mu(G_n) = \lim_{n \to \infty} \left(\mu(F_1) - \mu(F_n)\right) = \mu(F_1) - \lim_{n \to \infty} \mu(F_n)$$

Subtracting $\mu(F_1)$ (which is finite) from both sides yields the desired result.

(ii) Consider Lebesgue measure on $\mathbb{R}$. Let $F_n = [n, \infty)$. Then $m(F_n) = \infty$ for all $n \in \mathbb{N}$, so $\lim_{n \to \infty} m(F_n) = \infty$. However, $\cap_{n=1}^{\infty} F_n = \emptyset$, which has measure zero.

Exercise 16.7.

(a) Let $E = \{p\}$. Then $E = \cap_{n=1}^{\infty}\{(p - \frac{1}{n}, p + \frac{1}{n})\}_{n \in \mathbb{N}}$. Since E is an intersection of elements of $\mathcal{B}$, $E \in \mathcal{B}$. Further, since $\mu((p - \frac{1}{n}, p + \frac{1}{n})) = \frac{2}{n}$, by exercise 16.6 (b), $m(E) = \lim_{n \to \infty} \frac{2}{n} = 0$.

(b) If E is countable then E is a countable union of single points. So by part (a), since $\mathcal{B}$ is closed under countable unions and m is countably additive, $E \in \mathcal{B}$ and $m(E) = 0$.

Exercise 16.8.

Using the notation of section 2.44, the cantor set P can be written as $\cap_{n=1}^{\infty} E_n$ where $E_1 \supset E_2 \supset E_3 \supset \cdots$ and E_n is the union of 2^n intervals, each of length 3^{-n}. Therefore $m(E_n) = 2^n \cdot 3^{-n} = \left(\frac{2}{3}\right)^n$.
So applying exercise 16.6(b) we have $m(P) = \lim_{n \to \infty} \left(\frac{2}{3}\right)^n = 0$.

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Exercise 17.1.

(a) First note that any half open interval $(a, b]$ is equal to the intersection $\cap_{n=1}^{\infty}(a, b+\frac{1}{n})$ of open intervals. Since all open intervals are in the Borel σ-algebra and σ-algebras are closed under countable intersections, all half open intervals are in the Borel σ-algebra. So the σ-algebra generated by the half open intervals is contained in the Borel σ-algebra.

We have shown previously that any open set in $\mathbb{R}$ is a countable union open intervals (see exercise 4.11 (f)), so the Borel σ-algebra is generated by open intervals. Therefore, to show that the σ-algebra generated by the collection of all half open intervals contains the Borel σ-algebra is is sufficient to show that it contains all open intervals.

So fix any open interval $(a, b) \subset \mathbb{R}$. Then $(a, b) = \cup_{n=k}^{\infty}(a, b - \frac{1}{n})$ where $k \in \mathbb{N}$ is chosen such that $\frac{1}{k} < (b - a)$. Since σ-algebras are closed under countable unions, (a, b) is in the σ-algebra generated by the collection of all half open intervals.

(b) Since half rays are open sets and open sets generate the Borel σ-algebra, the σ-algebra generated by the collection of half rays is contained in the Borel σ-algebra.

For the reverse containment, by part (a) it is sufficient to show that the σ-algebra generated by the collection of all half rays contains all half-open intervals. Fix any half-open interval $(a, b]$. Then

$$(a, b] = \{x \in \mathbb{R} : x > a\} \cap \{x \in \mathbb{R} : x \le b\} = \{x \in \mathbb{R} : x > a\} \cap (\mathbb{R} \backslash \{x \in \mathbb{R} : x > b\})$$

Since σ-algebras are closed under complements and at most countable intersections, $(a, b]$ is in the σ-algebra generated by the collection of all half rays.

Exercise 17.2.

(a) By theorem 11.15, to show that $f(x)$ is measurable, it is sufficient to show that $f^{-1}([-\infty, a))$ is measurable for every $a \in \mathbb{R}$. So fix $a \in \mathbb{R}$.

I claim that $f^{-1}([-\infty, a)) = \cup_{n=1}^{\infty} f_n^{-1}([-\infty, a))$.

$$\begin{aligned} x \in f^{-1}([-\infty, a)) &\Leftrightarrow f(x) < a \\ &\Leftrightarrow \inf_n f_n(x) < a \\ &\Leftrightarrow \exists m \in \mathbb{N} \text{ such that } f_m(x) < a \\ &\Leftrightarrow \exists m \in \mathbb{N} \text{ such that } x \in f_m^{-1}([-\infty, a)) \\ &\Leftrightarrow x \in \cup_{n=1}^{\infty} f_n^{-1}([-\infty, a)) \end{aligned}$$

Since the f_n are measurable, by theorem 11.15 the sets $f_n^{-1}([-\infty, a))$ are measurable. Then since countable unions of measurable sets are measurable, $f^{-1}([-\infty, a)$ is measurable, as desired.

A similar argument proves the result for $F(x)$.

(b) Applying part (a) we know that for $m \in \mathbb{N}$, $\inf_{n\geq m} f_n$ is a measurable function. Then applying part (a) again we have that $\liminf_{n\to\infty} f_n = \sup_m \left(\inf_{n\geq m} f_n\right)$ is measurable. The proof for $\limsup_{n\to\infty} f_n$ is handled analogously.

Exercise 17.3.

We will prove this result by showing that $f_A^{-1}((a,\infty])$ is measurable for every $a \in \mathbb{R}$.
For $a \geq A$, $f_A^{-1}((a,\infty]) = \emptyset$ which is always measurable.
For $-A \leq a < A$, $f_A^{-1}((a,\infty]) = f^{-1}((a,\infty])$, which is measurable because f is a measurable function.
For $a < A$, $f_A^{-1}((a,\infty]) = \mathbb{R}$, which is always measurable.

Exercise 17.4.

(a) We verify the three properties of a σ-algebra:

(i) $f^{-1}(\mathbb{R}) = \mathbb{R}$, and since $\mathcal{M}$ is a σ-algebra, $\mathbb{R} \in \mathcal{M}$. Therefore $\mathbb{R} \in \mathcal{Y}$.

(ii) Let $E \in \mathcal{Y}$. Then

$$x \in \left(f^{-1}(E)\right)^c \iff x \notin f^{-1}(E) \iff f(x) \notin E \iff f(x) \in E^c \iff x \in f^{-1}(E^c)$$

Therefore $\left(f^{-1}(E)\right)^c = f^{-1}(E^c)$. Since $E \in \mathcal{Y}$, $f^{-1}(E) \in \mathcal{M}$. Then since $\mathcal{M}$ is a σ-algebra, $\left(f^{-1}(E)\right)^c \in \mathcal{M}$. Therefore $f^{-1}(E^c) \in \mathcal{M}$, so $E^c \in \mathcal{Y}$, as desired.

(iii) Let $\{E_n\}_{n=1}^\infty \subseteq \mathcal{Y}$. Then

$$x \in \cup_{n=1}^\infty f^{-1}(E_n) \iff \exists m \text{ s.t. } x \in f^{-1}(E_m) \iff x \in f^{-1}\left(\cup_{n=1}^\infty E_n\right)$$

So $\cup_{n=1}^\infty f^{-1}(E_n) = f^{-1}\left(\cup_{n=1}^\infty E_n\right)$. Since the $E_n \in \mathcal{Y}$, the $f^{-1}(E_n) \in \mathcal{M}$. Then since $\mathcal{M}$ is a σ-algebra, $\cup_{n=1}^\infty f^{-1}(E_n) \in \mathcal{M}$. Therefore $f^{-1}\left(\cup_{n=1}^\infty E_n\right) \in \mathcal{M}$, so $\cup_{n=1}^\infty E_n \in \mathcal{Y}$, as desired.

(b) Using the notation of part (a), from the definition of $\mathcal{A}$ we know that $\mathcal{A} \subseteq \mathcal{Y}$. Since $\sigma(\mathcal{A})$ is the smallest σ-algebra containing $\mathcal{A}$ and $\mathcal{Y}$ is a σ-algebra containing $\mathcal{A}$ we know that $\sigma(\mathcal{A}) \subseteq \mathcal{Y}$. Then by the definition of $\mathcal{Y}$, for every $F \in \sigma(\mathcal{A}) \subseteq \mathcal{Y}$ we know $f^{-1}(F) \in \mathcal{M}$.

Exercise 17.5.

(a) Any non-negative simple function ψ such that $\psi \leq \phi$ can be non-zero only on subsets of $\mathbb{Q}$. Since $m(\mathbb{Q}) = 0$, the measure of any subset of $\mathbb{Q}$ is also zero. Therefore $I_\mathbb{R}(\psi) = 0$. So taking the supremum of $I_\mathbb{R}(\psi)$ over all such ψ we have that $\int \phi dm = 0$.

(b) Let $\phi_\alpha = \alpha \cdot \chi_{E_\alpha}$. Then ϕ_α is a simple function with $\phi_\alpha \leq f$. So

$$\int f dm \geq I_\mathbb{R}(\phi_\alpha) = \alpha \cdot m(E_\alpha)$$

Dividing by α gives the desired inequality.

(c) $\Rightarrow$ Assume that $\int f dm = 0$. Using the notation of part (b) we have

$$x \in A \iff f(x) > 0 \iff \exists m \in \mathbb{N} \text{ s.t. } f(x) > 1/m \iff \exists m \in \mathbb{N} \text{ s.t. } x \in E_{1/m} \iff x \in \cup_{n=1}^{\infty} E_{1/n}$$

Therefore $A = \cup_{n=1}^{\infty} E_{1/n}$. By part (b), $m(E_{1/n}) = 0$ for all $n \in \mathbb{N}$. Therefore $m(A) \le \sum_{n=1}^{\infty} m(A_n) = 0$. Since $m(A)$ must be non-negative this implies $m(A) = 0$.

$\Leftarrow$ Assume that $m(A) = 0$. Let ϕ be any simple function such that $0 \le \phi \le f$. Then ϕ must be zero on A^c. Therefore, $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$ where there is one $j \in \{1, 2, \ldots, n\}$ such that $a_j = 0$ and $A^c \subseteq E_j$. Then for $i \neq j$, $E_j \subseteq A$, so $m(E_j) = 0$. Therefore $\int \phi \, dm = 0$, which implies that $\int f dm = 0$.

Exercise 17.6.
Let $f \in M^+(\mathbb{R}, \mathcal{L})$. For $n \in \mathbb{N}$ and $k = 0, 1, 2, \ldots, n2^n - 1$ define

$$E_{n,k} = f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right),$$

For $k = n2^n$ define $E_{n,k} = f^{-1}([n, \infty])$. Then for $n \in \mathbb{N}$ let

$$\phi_n = \sum_{k=0}^{n2^n} \frac{k}{2^n} \cdot \chi_{E_{n,k}}$$

Exercise 1(a) combined with the fact that pre-images of measurable sets under measurable functions are measurable shows that the sets $E_{n,k} \in \mathcal{L}$. For any fixed $n \in \mathbb{N}$, since the intervals $\{[k2^{-n}, (k+1)2^{-n})\}_{k=0}^{n2^n-1} \cup \{[n, \infty]\}$ are disjoint and cover the entire range of f, the sets $\{E_{n,k}\}_{k=1}^{n2^n}$ will be disjoint and cover all of $\mathbb{R}$. For any fixed n each of the coefficients $k2^{-n}$ are distinct for different k values. If any of the $E_{n,k}$ are empty we can remove them from the defining sum for ϕ_n without changing the value of ϕ_n. Therefore, for each $n \in \mathbb{N}$, ϕ_n can be written as a finite linear combination of characteristic functions of disjoint non-empty measurable sets with distinct coefficients, so ϕ_n is a simple function.

(a) Fix any $x \in \mathbb{R}$ and any $n \in \mathbb{N}$. We consider two cases.

- $x \in E_{n,k}$ for some $k \in \{0, 1, \ldots n2^n - 1\}$.
  Since $[k2^{-n}, (k+1)2^{-n}) = [2k2^{-(n+1)}, (2k+1)2^{-(n+1)}) \cup [(2k+1)2^{-(n+1)}, (2k+2)2^{-(n+1)})$ we know that $x \in E_{n+1,2k} \cup E_{n+1,2k+1}$.
  So $\phi_{n+1}(x) \in \{2k2^{-(n+1)}, (2k+1)2^{-(n+1)}\} = \{k2^{-n}, (k+1/2)2^{-n}\}$.
  Therefore $\phi_n(x) = k2^{-n} \le \phi_{n+1}(x)$.
- $x \in E_{n,n2^n}$. If $f(x) = \infty$ then $\phi_n(x) = n < n+1 = \phi_{n+1}(x)$. Otherwise $x \in [k2^{-n}, (k+1)2^{-n})$ for some integer $k \ge n2^n$.
  So by the argument of the above case $x \in [l2^{-(n+1)}, (l+1)2^{-(n+1)})$ for $l \in \{2k, 2k+1\}$.
  If $l \ge (n+1)2^{-(n+1)}$ then $\phi_{n+1}(x) = (n+1) > n = \phi_n(x)$.
  If $l < (n+1)2^{-(n+1)}$ then $\phi_{n+1}(x) = l2^{-(n+1)} \ge 2k2^{-(n+1)} = k2^{-n} \ge n = \phi_n(x)$.

So in all cases we have $\phi_n(x) \leq \phi_{n+1}(x)$ for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$. Since all of the coefficients $k2^{-n}$ are non-negative, all of the ϕ_n are non-negative. So $0 \leq \phi_n \leq \phi_{n+1}$ for all $n \in \mathbb{N}$.

(b) Fix any point $x \in X$ and any $\epsilon > 0$. If $f(x) = \infty$ then $\phi_n(x) = n$ for all $n \in \mathbb{N}$, so $\lim_{n\to\infty} \phi_n(x) = f(x)$. If $f(x) < \infty$, fix $N \in \mathbb{N}$ such that $f(x) < N$ and $\frac{1}{2^N} < \epsilon$. Now fix any $m > N$. There exists $j_m \in \{0, 1, 2, \ldots, m2^m - 1\}$ such that $x \in E_{m,j_m}$. Then $\phi_m(x) = j_m 2^{-m}$ and $f(x) \in [j_m 2^{-m}, (j_m + 1)2^{-m})$. Therefore $|f(x) - \phi_n(x)| \leq (j_m + 1)2^{-m} - j_m 2^{-m} = 2^{-m} < \epsilon$. Since $x \in X$ and $\epsilon > 0$ were arbitrary, this shows that $\phi_n(x)$ converges point-wise to $f(x)$.

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Exercise 18.1.

$\Rightarrow$ If f is measurable then by theorem 11.15 the function $-f$ is also measurable. The constant function 0 is also measurable since the pre-image of any set is either $\mathbb{R}$ or $\emptyset$. So by exercise 17.2 (a), since the supremum of measurable functions is measurable, f^+ and f^- are both measurable.
$\Leftarrow$ Note that for $x \in \mathbb{R}$ such that $f(x) \geq 0$, $f(x) = f^+(x)$ and $f^-(x) = 0$ while for $f(x) < 0$, $f(x) = -f^-(x)$ and $f^+(x) = 0$. Therefore $f(x) = f^+(x) + (-f^-(x))$ for all $x \in \mathbb{R}$. Since sums of measurable functions are measurable (Theorem 11.18) we have that f is measurable.

Exercise 18.2.

Define $f_n(x) = \chi_{[0,n]}$. Since $[0, n]$ is a measurable set for all $n \in \mathbb{N}$, f_n is measurable for all $n \in \mathbb{N}$. Since the f_n are characteristic functions they are uniformly bounded by 1. For $x < 0$ we have $f_n(x) = 0$ for all $n \in \mathbb{N}$, so $f_n(x)$ is monotone increasing to zero. For $x \geq 0$, $f_n(x) = 0$ for $n < x$ and then $f_n(x) = 1$ for $n \geq x$. Therefore $f_n(x)$ is monotone increasing to one. So the functions f_n are monotone increasing to $\chi_{[0,\infty)}$, which as a characteristic function of a measurable set is measurable.
However, $\int f_n dm = 1 \cdot \mu([0, n]) = n$, while $\int f dm = 1 \cdot \mu([0, \infty)) = \infty$.
In this case the hypotheses of the monotone convergence theorem are met and we have $\lim_{n\to\infty} \int f_n dm = \lim_{n\to\infty} n = \infty = \int f dm$

Exercise 18.3.

Define $f_n(x) = 1/n$, the constant function. Then $|f_n(x) - f(x)| = 1/n$ for all $x \in \mathbb{R}$, $n \in \mathbb{N}$, so the f_n converge uniformly to f. Since $\{1/n\}$ is monotone decreasing the f_n are all monotone decreasing. However, $\int f_n dm = \frac{1}{n} \cdot \mu(\mathbb{R}) = \infty$ for all $n \in \mathbb{N}$.

Exercise 18.4.

Let $E_n = \{x \in \mathbb{R} : f(x) \geq n\}$. Then $n\chi_{E_n}$ is a simple function that is less than or equal to f. So for all $n \in \mathbb{N}$ we have

$$\infty > \int f dm \geq \int n\chi_{E_n} = n \cdot m(E_n)$$

Therefore $m(E_n) \leq \frac{1}{n} \int f dm$ for all $n \in \mathbb{N}$. So $\lim_{n\to\infty} m(E_n) = 0$.
Also note that $x \in E \iff f(x) = \infty \iff f(x) \geq n \;\; \forall n \in \mathbb{N} \iff x \in \cap_{n=1}^{\infty} E_n$ Therefore $E = \cap_{n=1}^{\infty} E_n$. Then applying exercise 16.6 (b) we have that $m(E) = m(\cap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} m(E_n) = 0$.

Exercise 18.5.

(a) If $f(x) = 0$ almost everywhere, then the set $A := \{x \in \mathbb{R} : f(x) \neq 0\}$ has measure zero. Since the functions f^+ and f^- are zero whenever f is zero, the sets $A^+ := \{x \in \mathbb{R} : f^+(x) \neq 0\}$ and $A^- := \{x \in \mathbb{R} : f^-(x) \neq 0\}$ are both contained in A and hence both have measure zero. So by exercise 17.5 (c), f^+ and f^- are both integrable with integral zero. Therefore $\int f dm = 0$.

(b) If $f(x) = g(x)$ a.e. on $\mathbb{R}$, then the function $g - f = 0$ a.e. on $\mathbb{R}$. Since f and g are measurable, the function $g - f$ is also measurable. Therefore by part (a), $0 = \int (g - f)dm$. Since f is integrable we can add $\int f dm$ to both sides of the above equality. Applying Theorem 11.29 then yields

$$\int f dm = \int (g - f)dm + \int f dm = \int (g - f + f)dm = \int g dm$$

Exercise 18.6.

Let $\psi(x) = \sum_{n=1}^{\infty} |f_n(x)|$. Then by Theorem 11.30 (this uses monotone convergence) we have

$$\int \psi dm = \sum_{n=1}^{\infty} \int |f_n| dm < \infty$$

Then by exercise 18.4 we know that the set $E := \{x \in \mathbb{R} : \psi(x) = \infty\}$ has measure zero. For $x \in E^c$, $\psi(x) < \infty$, so the series $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely. Therefore $\sum_{n=1}^{\infty} f_n(x)$ converges to some function f on E^c.

Let $g_n = \sum_{m=1}^{n} f_m$. Then $|g_n(x)| = |\sum_{m=1}^{n} f_m(x)| \leq \sum_{m=1}^{n} |f_m(x)| \leq \psi(x)$ for all $x \in E^c$, and $g_n(x) \to f(x)$ for all $x \in E^c$. So by Lebesgue's dominated convergence theorem we have

$$\int_{E^c} f dm = \lim_{n\to\infty} \int_{E^c} g_n dm = \lim_{n\to\infty} \int_{E^c} \sum_{m=1}^{n} f_n dm = \lim_{n\to\infty} \sum_{m=1}^{n} \int_{E^c} f_n dm = \sum_{n=1}^{\infty} \int_{E^c} f_n dm$$

In the third equality we have used Theorem 11.29 applied to the finite sum.

Since $m(E) = 0$, these results extend to all of $\mathbb{R}$

$$\int f dm = \int_{E^c} f dm = \sum_{n=1}^{\infty} \int_{E^c} f_n dm = \sum_{n=1}^{\infty} \int f_n dm$$

Since $\sum_{n=1}^{\infty} \int f_n dm \leq \sum_{n=1}^{\infty} \int |f_n| dm < \infty$, this also shows that f is integrable.