Pricing Call Options Under the Binomial Model

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Abstract

A call option gives you the option to buy the stock for a fixed price at a given future date. We want to find a mathematical model so that we can calculate the value of the call option at any time step from now to the maturity. A CDF gives us more information about the call option, so we find a sequence of CDF's based on the time steps and use characteristic equations and Levy's Continuity Theorem to find that these CDF's converge to a CDF after some rescaling. Finally, we use this to derive an equation for the price of the call option today as the number of time steps converges to infinity.

1 Introduction

A call option gives you the option to buy the stock for a certain fixed price at a given future date, called the maturity. The decision to buy the option is a gamble. Essentially, the price of the call option is a function of the stock price and it tells us how much to pay for the profit that we can receive. We use a probabilistic model to see what might happen to the stock until the maturity. Since the call option is a function of the stock price, we can model the stock price to give us more information about the call option.

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Suppose we want to know what the price of the stock will be one month from now and assume that the price of the stock changes four times during this month. From these assumptions, we can find all the possibilites of the stock price in one month. The first time it changes in this month, the stock will go up or down, then from these two possibilites the stock price will go up or down again. We have four possibilities now, and for each of these possibilities we also have two possibilities. There is also a probability associated with whether the stock will go up or down as well.

We model the future to know what to do today. We create a model that satisfies these properties. Once we find this model, we can use it to try and find out what to do now by knowing what can happen until the maturity with a stock that changes n times in that time period. We propose to look at the behavior of the model as the number of intermediate steps converges to infinity. And, with this mathematical fortune telling insight, we can compare the price of the call option we found with our model with the current status of the stock and gamble less ignorantly.

2 Creating the Model

Figure 1 gives a visual representation of how the model for the stock is created.

Let Ω be the state space or the set of all paths. Let S_0 denote the initial stock price, and let S_T be the random variable representing the value of the stock at the maturity or the last step. These two random variables are at fixed times, and in the case of Figure 1, this example has four steps from the inital to maturity. So, for this particular case we denote S_0 as S_0^4 and S_T as S_4^4. We formalize this in the following way.

Definition 1 *Let j be the time step such that $j = 0, 1, 2, ..., n$, where n is the number of time steps until maturity, denote*
S_j^n *- the value of the stock at time j, and*
S_n^n *- the value of the stock at maturity of a model with n steps.*

Now, we introduce call options and the profit associated to it in order to relate these to the model we have created.

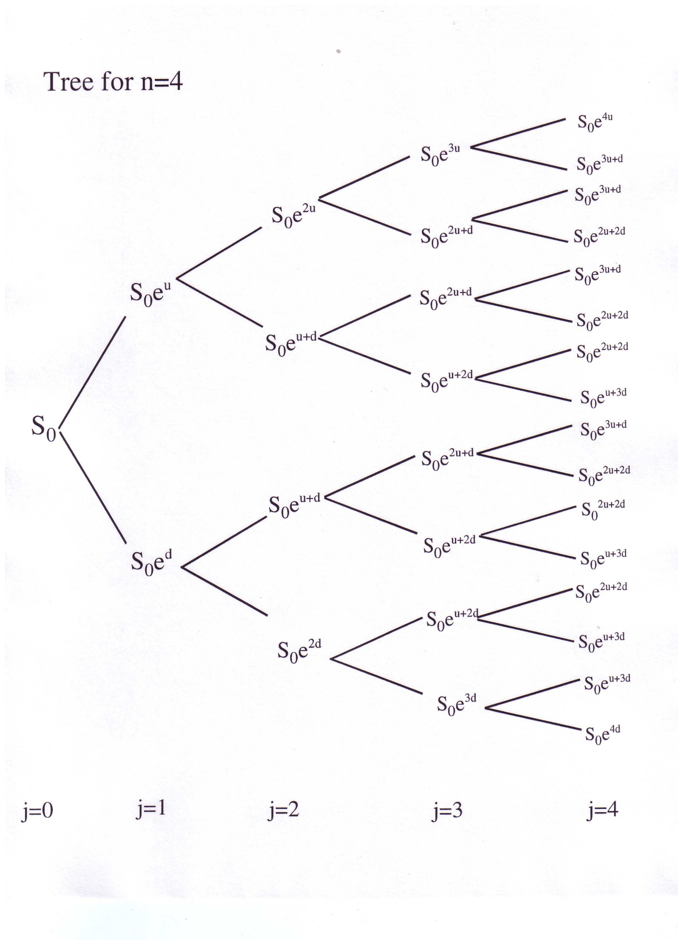

Figure 1: Binomial Model for $n = 4$

Definition 2 (Call option of strike K, maturity T) *A call option of strike K and maturity T, denoted C, is a financial contract which gives you the* ***option*** *to buy the stock at price K at time T.*

The following definition follows from the previous one:

Definition 3 (Profit) *The profit of the call option is the value of the call option at maturity T, denoted by*

$$C_T = \max(S_T - K, 0).$$

So, C_T is a function of S_T, which defines it as a random variable as well, and we formalize it similarly.

Definition 4 *Let j be the time step such that $j = 0, 1, 2, ..., n$, where n is the number of time steps until maturity, denote*
C_j^n *- the value of the call option at time j,*
C_n^n *- the value of the call option at maturity.*

Since we have sequences of random variables, C and S are stochastic processes. Hence, we want to associate a probability with these events and a probability measure for it.

The expected value, or expectation, can be described as a weighted sum in the discrete case. It can be found by taking the summation over the possible values of the random variable and multiplying them by the probability of each value occuring. We denote the expectation of a random variable X as

$$\mathbb{E}(X) = \sum_{x:f(x)>0} xf(x).$$

Definition 5 (Probability Measure, $\mathbb{Q}$) *We define*

$$\mathbb{Q} : \Omega \to [0, 1]$$

as the measure that satisfies $\mathbb{E}_{\mathbb{Q}}(S_{j+1}|S_j) = S_j$.

Definition 6 (Price of a Call Option) *We define the price of a call option at time j as $C_j^n = \mathbb{E}_{\mathbb{Q}}(C_T|S_j)$ for $j = 1, 2, ..., n$.*

Definition 5 implies that the value of the stock under the measure $\mathbb{Q}$ is equal to the expectation of the value of the stock at the beginning, and the value of the call option at maturity is equal on average to the value of the call option at the starting point.

We start by analyzing the base case, where $n = 1$, so that there is only one time step from the initial to maturity. To visualize this, imagine the tree in Figure 1 to stop growing after the initial two branches from S_0 (or cut off the tree after $j = 1$). From the initial date to maturity we have one step, so since we assume that the stock increases or decreases, let e^u be the parameter that increases the stock and e^d be the parameter that decreases the stock, where $d < 0 < u$. Now, we can look at the value of the stock at maturity as a function, i.e. $S_1^1 : \Omega \to \{S_0e^u, S_0e^d\}$. Let ω_1 be the path that ends at S_0e^u, and ω_2 be the path that ends at S_0e^d, so $\mathbb{Q}(\omega_1) = q$, which is the probability of the stock increasing, and $\mathbb{Q}(\omega_2) = (1-q)$, which is the probability of the stock decreasing, and can be represented in the following table.

S_1^1	Frequency	$\mathbb{Q}$
S_0e^u	1	q
S_0e^d	1	$(1-q)$

We know that C_1^1 is a function of S_1^1. And, we find what the initial price of the call option should be with $C_0 = \mathbb{E}_{\mathbb{Q}}C_1^1$.

Since we are interested in what will happen if we increase the number of time steps between the initial time and the maturity, we look at the next simple case for two steps, or $n = 2$. We can visualize this by cutting Figure 1 at $j = 2$.

Now, the stock has two places to rise or fall from the $j = 1$ step, so S_2^2 can take on four values. In general, S_n^n will have 2^n possible values or the tree paths will have 2^n destinations. Once again, if we rise we have the parameter e^u and fall is e^d, then when the stock falls from S_0e^u, it becomes S_0e^{u+d}. Also, when the stock rises from S_0e^d, it becomes S_0e^{u+d}, which gives us two different events with the same outcome.

But, can we verify that the probability of the stock price rising is q and falling is $(1-q)$ for any time step? We will verify this by how we defined $\mathbb{Q}$. We denote the probability of rising from S_0e^u as q_1 and for rising from S_0e^d as q_2. By using the definition of expectation we get the following equations:

$$S_0e^u = S_0e^{2u}q_1 + S_0e^{u+d}(1-q_1).$$
$$S_0e^d = S_0e^{u+d}q_2 + S_0e^{2d}(1-q_2).$$

If we solve for q_1 and q_2, we find their solution is

$$q_1 = \frac{1-e^d}{e^u - e^d} = q_2.$$

We would like to generalize this result for all n. Consider a particular branch for some step size n at position (j, m), where j is our time step and m is one of the particular branches for that time step. The next equation shows the generalization.

$$q_{jm} S_0 e^{((m+1)u+jd)} + (1 - q_{jm}) S_0 e^{mu+(j+1)d} = S_0 e^{mu+jd}$$

for $j = 1, ..., n$ and $m \leq 2^j$.

Solving for q_{jm} we obtain the following:

$$q_{jm} = \frac{1 - e^d}{e^u - e^d}.$$

Hence q is the same for all j and m.

Now, we can return to the case for $n = 2$. Since the probability to get $S_0 e^d$ is $(1 - q)$ and from there to get to $S_0 e^{u+d}$ is $(1 - q)q$ since it's the probability of decreasing and increasing, and these are independent events, we multiply. The following table illustrates this.

S_2^2	Frequency	$\mathbb{Q}$
$S_0 e^{2u}$	1	q^2
$S_0 e^{u+d}$	2	$2q(1-q)$
$S_0 e^{2d}$	1	$(1-q)^2$

For S_3^3, we obtain the following table:

S_3^3	Frequency	$\mathbb{Q}$
$S_0 e^{3u}$	1	q^3
$S_0 e^{2u+d}$	3	$3q^2(1-q)$
$S_0 e^{u+2d}$	3	$3q(1-q)^2$
$S_0 e^{3d}$	1	$(1-q)^3$

It is clear that this is a binomial model, so in general we have the following table:

S_n^n	Frequency	$\mathbb{Q}$
S_0e^{nu}	$\binom{n}{0}$	$\binom{n}{0}q^n$
$S_0e^{(n-1)u+1d}$	$\binom{n}{1}$	$\binom{n}{1}q^{n-1}(1-q)$
$S_0e^{(n-2)u+2d}$	$\binom{n}{2}$	$\binom{n}{2}q^{n-2}(1-q)^2$
$\vdots$	$\vdots$	$\vdots$
$S_0e^{(n-j)u+jd}$	$\binom{n}{j}$	$\binom{n}{j}q^{n-j}(1-q)^j$
$\vdots$	$\vdots$	$\vdots$
S_0e^{nd}	$\binom{n}{n}$	$\binom{n}{n}(1-q)^n$

Now, we have a simple way to quantify the model. Let us quantify C_0^n. So,

$$\begin{aligned} C_0^n &= \mathbb{E}_{\mathbb{Q}}C_T \\ &= \mathbb{E}_{\mathbb{Q}}\max(S_T - K, 0) \\ &= \sum_{i=0}^{n} q^{n-i}(1-q)^i \binom{n}{i} \max(S_0e^{(n-i)u+id} - K, 0). \end{aligned}$$

We have binomial coefficient $\binom{n}{i} = \frac{n!}{k!(n-k)!}$ and $\max(S_0e^{(n-i)u+id} - K, 0)$ as the profit made from the option. Clearly if $S_0e^{(n-i)u+id} - K \leq 0$, then no profit is accumulated.

Say we are interested in time steps other than the initial time, then the previous result can be extended to:

$$C_j^n = \sum_{i=0}^{n-j} q^{n-j-i}(1-q)^i \binom{n-j}{i} \max(S_j e^{(n-j-i)u+id} - K, 0).$$

This means we are looking at call options with initial price S_j^n and $n-j$ steps remaining.

Also, in the Stock Market, the stock prices change many times before maturity, so in our model we want to have a large number of n-steps. In this sense, the best model used to find C_0 would be to take this model for sufficiently large n or as $n \to \infty$.

So, with the model we find explicit formulas for the expectation, variance and CDF of S_T. Then we will be able to calculate and reach a nice result for these formulas as $n \to \infty$ or as the tree gets infinitely large to further perfect the model.

3 Probability for Probabilistic Model

3.1 About CDF

A cumulative distribution function, or CDF, is a function that describes distribution of a random variable, which we denote by F. So, $F(x) = \mathbb{Q}[X \leq x]$, where $\mathbb{Q}$ is the probability measure and X is a random variable. $F(x) = \mathbb{Q}[X \leq x]$ stands for the probability of the random value X taking on values less than or equal to x. It should be noted that all the values that the CDF takes on are between 0 and 1.

We continue by finding a general formula for the CDF of S_T. We see that the CDF for S_T is

$$F_n(x) = \begin{cases} 0 & \text{if } x < S_0 e^{nd} \\ \sum_{j=i}^{n} \binom{n}{j} q^{n-j}(1-q)^j & \text{if } S_0 e^{(n-i)u+id} \geq x \geq S_0 e^{(n-(i+1))u+(i+1)d} \\ 1 & \text{if } x > S_0 e^{nu} \end{cases}$$

To simplify our computations, we notice that $\log(\frac{S_T}{S_0})$ represents the return on the stock. By using $\log(\frac{S_T}{S_0})$ instead of S_T, we will be able to compute later calculations with more ease. We find that the cumulative distribution function of $\log(\frac{S_T}{S_0})$ will be

$$F_n(x) = \begin{cases} 0 & \text{if } x < nd \\ \sum_{j=i}^{n} \binom{n}{j} q^{n-j}(1-q)^j & \text{if } (n-i)u + id \geq x \geq (n-(i+1))u + (i+1)d \\ 1 & \text{if } x > nu \end{cases}$$

3.2 Expectation and Variance of $\log \frac{S_T}{S_0}$

To find the expectation of $\log \frac{S_T}{S_0}$, we multiply each value that $\log \frac{S_T}{S_0}$ takes, which are of the form $[(n-i)u + id]$ for $i = 0, 1, ..., n$, by the probability of each one occuring, (which are of the binomial form), and then add these values together. So,

$$\mathbb{E}\left(\log\frac{S_T}{S_0}\right) = \sum_{i=0}^{n}\binom{n}{i}q^{n-i}(1-q)^i[(n-i)u+id].$$

We then use the property $\sum_{i=0}^{n}\binom{n}{i}q^{n-i}(1-q)^i i = n(1-q)$ to reduce our equation. We find that

$$\mathbb{E}\left(\log\frac{S_T}{S_0}\right) = -n[(u-d)q+d].$$

Definition 7 *The variance of a random variable X is the measurement of how its collective values will vary, and is given by the equation*

$$VarX = \mathbb{E}[(X)^2] - [\mathbb{E}(X)]^2.$$

By the definition of variance,

$$\begin{aligned} Var(\log\frac{S_T}{S_0}) &= \mathbb{E}((\log\frac{S_T}{S_0})^2) - (\mathbb{E}(\log\frac{S_T}{S_0}))^2 \\ &= \sum_{i=0}^{n}\binom{n}{i}q^{n-i}(1-q)^i[(n-i)+id]^2 - (n[u-d]q+d)^2. \end{aligned}$$

By much distributing and breaking the summation into parts, we come to our final equation for the variance:

$$\begin{aligned} Var(\log\frac{S_T}{S_0}) &= u^2nq - u^2nq^2 - 2udnq + 2udnq^2 - d^2nq^2 + d^2qn \\ &= n(1-q)q(d-u)^2. \end{aligned}$$

4 Rescaling u and Comparing our old and new CDF

If we plot $F_n(x)$ we obtain Figure 2 for $n = 5, 50, 100, 500, 1000, 2000, 4000, 6000, 8000$.

From this figure, as the value of n increases, we can see that values that F_n takes on begin to decrease, so it can be shown that this

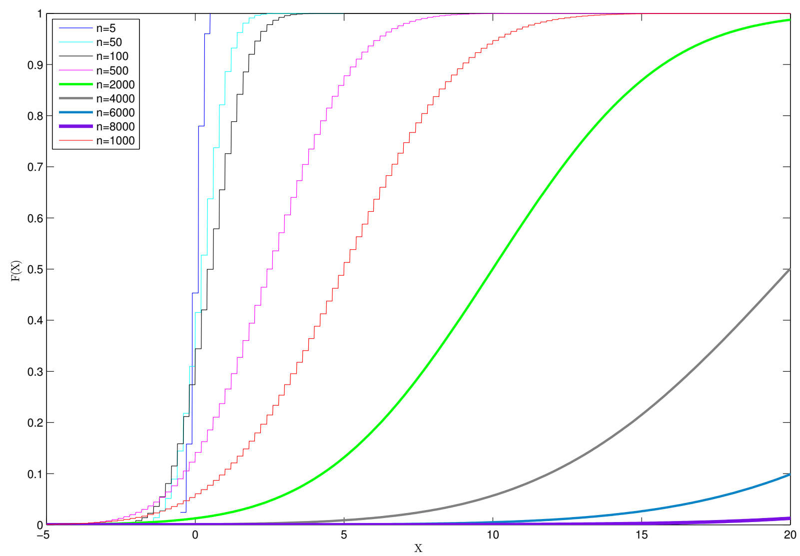

Figure 2: Plot of CDF Before Rescaling u

sequence of functions will converge pointwise to $F(x) = 0\ \forall x$.

In order to find what $\{F_n\}$ converges to, we first look at what $\mathbb{E}\left(\log\left(\frac{S_T}{S_0}\right)\right)$ and $Var\left(\log\left(\frac{S_T}{S_0}\right)\right)$ converge to. We see that for the expectation

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\left(\log\left(\frac{S_T}{S_0}\right)\right) &= \lim_{n\to\infty} n\left[(u-d)\,q+d\right] \\ &= \infty\end{aligned}$$

and for the variance

$$\begin{aligned}\lim_{n\to\infty} Var\left(\log\left(\frac{S_T}{S_0}\right)\right) &= \lim_{n\to\infty} n(1-q)q(d-u)^2 \\ &= \infty.\end{aligned}$$

These two facts suggest that $\{F_n\}$ does not converge to a CDF as $n \to \infty$. This is not the result we want, since we would like the model to work for large n, which means that as $n \to \infty$ we want $\{F_n\}$ to converge to a CDF. In other words, with the current u and d, we do not get a reasonable model for a stock in the limit. So we make the decision to rescale u. We first tried to rescale u to be $\frac{u}{n}$, but then we found that for the expectation

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\left(\log\left(\frac{S_T}{S_0}\right)\right) &= \lim_{n\to\infty} n\left[\left(\frac{u}{n}+\frac{u}{n}\right)q-\frac{u}{n}\right] \\ &= 0\end{aligned}$$

and for the variance

$$\begin{aligned}\lim_{n\to\infty} Var\left(\log\left(\frac{S_T}{S_0}\right)\right) &= \lim_{n\to\infty} n\left(1-q\right)q\left(-\frac{u}{n}-\frac{u}{n}\right)^2 \\ &= 0.\end{aligned}$$

However, this was not a useful result, since it is unrealistic to have a zero variance. So, once again we do not obtain a CDF for the model as $n\to\infty$. We then let $U_n=\frac{u}{\sqrt{n}}$ and define $d=-U_n$. So, for the expectation

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\left(\log\left(\frac{S_T}{S_0}\right)\right) &= \lim_{n\to\infty} n\left[\left(U_n-(-U_n)\right)q+(-U_n)\right] \\ &= \lim_{n\to\infty} n\left[\left(\frac{u}{\sqrt{n}}+\frac{u}{\sqrt{n}}\right)\left(\frac{1}{e^{\frac{u}{\sqrt{n}}}+1}\right)-\frac{u}{\sqrt{n}}\right] \\ &= \lim_{n\to\infty} n\frac{u}{\sqrt{n}}\left(\frac{2}{e^{\frac{u}{\sqrt{n}}}+1}-1\right) \\ &= \lim_{n\to\infty} u\sqrt{n}\left(\frac{2}{e^{\frac{u}{\sqrt{n}}}+1}-1\right) \\ &= \lim_{n\to\infty} u\sqrt{n}\left(\frac{2-e^{\frac{u}{\sqrt{n}}}-1}{e^{\frac{u}{\sqrt{n}}}+1}\right) \\ &= \lim_{n\to\infty} u\sqrt{n}\left(\frac{1-e^{\frac{u}{\sqrt{n}}}}{e^{\frac{u}{\sqrt{n}}}+1}\right).\end{aligned}$$

Now by Taylor's Theorem, we know that $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{e^{\xi}x^4}{4!}$ where $\xi\in(0,x)$, so

$$\begin{aligned}
\lim_{n\to\infty} u\sqrt{n}\left(\frac{1-e^{\frac{u}{\sqrt{n}}}}{e^{\frac{u}{\sqrt{n}}}+1}\right) &= \lim_{n\to\infty} u\sqrt{n}\left(\frac{1-\left(1+\frac{u}{\sqrt{n}}+\frac{u^2}{2!n}+\frac{u^3}{3!n^{\frac{3}{2}}}+\frac{e^{\xi}u^4}{4!n^2}\right)}{1+1+\frac{u}{\sqrt{n}}+\frac{u^2}{2!n}+\frac{u^3}{3!n^{\frac{3}{2}}}+\frac{e^{\xi}u^4}{4!n^2}}\right) \\
&= \lim_{n\to\infty}\left(\frac{-u^2-\frac{u^3}{2!\sqrt{n}}-\frac{u^4}{3!n}-\frac{e^{\xi}u^4}{4!n^{\frac{3}{2}}}}{2+\frac{u}{\sqrt{n}}+\frac{u^2}{2!n}+\frac{u^3}{3!n^{\frac{3}{2}}}+\frac{e^{\xi}u^4}{4!n^{\frac{3}{2}}}}\right) \\
&= -\frac{u^2}{2},
\end{aligned}$$

and for the variance

$$\begin{aligned}
\lim_{n\to\infty} Var\left(\log\left(\frac{S_T}{S_0}\right)\right) &= \lim_{n\to\infty} n\,(1-q)\,q\,(-U_n-U_n)^2 \\
&= \lim_{n\to\infty} n\left(1-\frac{1}{e^{\frac{u}{\sqrt{n}}}+1}\right)\frac{1}{e^{\frac{u}{\sqrt{n}}}+1}\left(-\frac{u}{\sqrt{n}}-\frac{u}{\sqrt{n}}\right)^2 \\
&= \lim_{n\to\infty} n\left(1-\frac{1}{e^{\frac{u}{\sqrt{n}}}+1}\right)\frac{1}{e^{\frac{u}{\sqrt{n}}}+1}\frac{4u^2}{n} \\
&= \lim_{n\to\infty}\left(\frac{e^{\frac{u}{\sqrt{n}}}+1-1}{e^{\frac{u}{\sqrt{n}}}+1}\right)\left(\frac{1}{e^{\frac{u}{\sqrt{n}}}+1}\right)4u^2 \\
&= u^2.
\end{aligned}$$

This is a desirable result since for any u that is given to us we can use the model. And, a finite variance will provide us with a sequence of CDF's which converge to a CDF, which will be shown in the next section. Figure 3 illustrates this property of the sequence of CDF's once we rescaled u.

5 F_n approaches a CDF as $n \to \infty$

In this section we prove the convergence of a sequence of CDF's.

Here's our main theorem:

Theorem 1 *Let F be the CDF of a normal with mean $\mu = -\frac{1}{2}u^2$ and variance $\sigma^2 = u^2$; then $F_n \to F$ as $n \to \infty$.*

Before we prove our main theorem, we state the following known result:

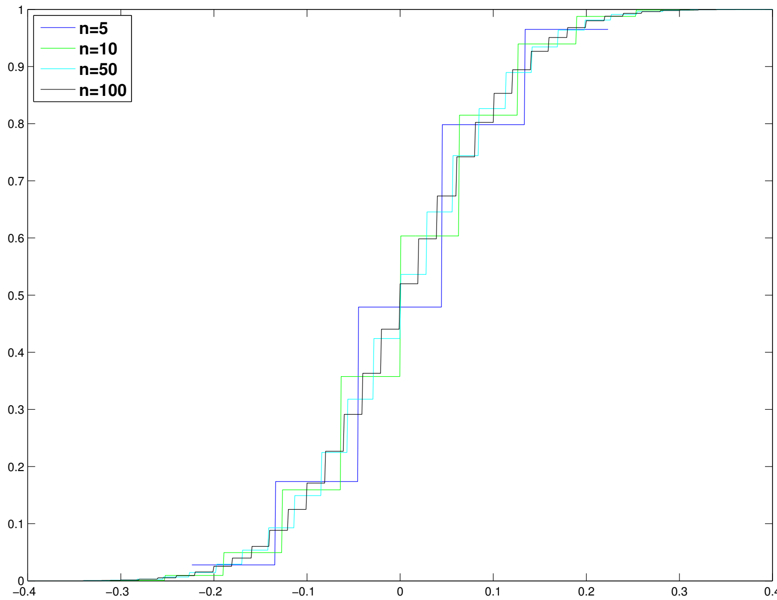

Figure 3: Plot of CDF After Rescaling u

Theorem 2 (Levy's Continuity Theorem) *Let (X_n) be a sequence of random variables with CDF F_n and characteristic function ϕ_n. If $\phi_n \to \phi$ and if in addition ϕ is continuous at 0, then there exists a CDF F such that $F_n \to F$.*

Proof. (of Theorem 1) Our goal is to use the idea of characteristic functions to say something about the convergence of the CDF. For more feasibility, we use the characteristic function over the logarithm of the random variable S_T to be

$$\phi_n(t) \equiv \mathbb{E}e^{it\log(\frac{S_T}{S_0})}.$$

We are first interested in showing that

$$\phi_n(t) = ((1-q)e^{-itU_n} + qe^{itU_n})^n$$

Here $U_n = \frac{u}{\sqrt{n}}$ is our rescaled value.

Taking the expectation in the initial expression we have

$$\sum_{j=0}^{n} \binom{n}{j} q^{n-j}(1-q)^j e^{it[(n-j)U_n - jU_n]}.$$

We rearrange and factor out indices to get

$$e^{itnU_n} q^n \sum_{j=0}^{n} \binom{n}{j} (\frac{1-q}{q})^j e^{j(-itU_n - itU_n)}.$$

Then collecting terms under j we have

$$e^{itnU_n} q^n \sum_{j=0}^{n} \binom{n}{j} [(\frac{1-q}{q}) e^{-itU_n - itU_n}]^j.$$

Setting $x = [(\frac{1-q}{q})e^{-itU_n - itU_n}]^j$, it follows that

$$\phi_n(t) = e^{itnU_n} q^n \sum_{j=0}^{n} \binom{n}{j} (x)^j 1^{n-j}.$$

Recall the following identity:

$$\sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j} = (x+y)^n.$$

We now use this to solve the sum and obtain

$$\phi_n(t) = e^{itnU_n} q^n (\frac{(1-q)e^{-itU_n - itU_n} + q}{q})^n.$$

Finally, after bringing the $q's$ under the exponent of n we conclude that

$$\phi_n(t) = ((1-q)e^{itd_n} + qe^{itU_n})^n.$$

Define $\psi_n(t) = (1-q)e^{-itU_n} + qe^{itU_n}$. Keeping in mind that we want to know the limiting value of the above expression, let's re-write the term under the exponent as a Taylor Series centered at zero with error term as follows

$$\psi_n(t) = \psi_n(0) + \psi_n'(0)t + \psi_n''(0)\frac{t^2}{2!} + \psi_n'''(\xi)\frac{t^3}{3!} \quad \text{for some } \xi \in (0,t)$$

Since $\psi_n(t) = (1-q)e^{-itU_n} + qe^{itU_n}$, we can list its derivative explicitly:

$$\begin{aligned}
\psi_n'(t) &= -\frac{iu}{\sqrt{n}}e^{-it\frac{u}{\sqrt{n}}} + q\frac{iu}{\sqrt{n}}e^{-it\frac{u}{\sqrt{n}}} + q\frac{iu}{\sqrt{n}}e^{it\frac{u}{\sqrt{n}}} \\
\psi_n''(t) &= \frac{i^2u^2}{n}e^{-it\frac{u}{\sqrt{n}}} - q\frac{i^2u^2}{n}e^{-it\frac{u}{\sqrt{n}}} + q\frac{i^2u^2}{n}e^{it\frac{u}{\sqrt{n}}} \\
\psi_n'''(t) &= -\frac{i^3u^3}{n^{\frac{3}{2}}}e^{-it\frac{u}{\sqrt{n}}} + q\frac{i^3u^3}{n^{\frac{3}{2}}}e^{-it\frac{u}{\sqrt{n}}} + q\frac{i^3u^3}{n^{\frac{3}{2}}}e^{it\frac{u}{\sqrt{n}}}.
\end{aligned}$$

Noting that the error term comes directly from Taylor's Remainder Theorem (i.e. $R_n = \frac{(t-t_0)^{n+1}}{(n+1)!}f^{n+1}(\xi)$), we would like this derivative to be bounded by a constant. Computing the third derivative at ξ we have

$$\begin{aligned}
|\psi'''(\xi)| &= |\frac{i^3u^3}{n^{\frac{3}{2}}}[e^{-i\xi\frac{u}{\sqrt{n}}} + qe^{-i\xi\frac{u}{\sqrt{n}}} + qe^{i\xi\frac{u}{\sqrt{n}}}]| \\
&\leq \frac{|u^3|}{n^{\frac{3}{2}}}[|e^{-i\xi\frac{u}{\sqrt{n}}}| + |q||e^{-i\xi\frac{u}{\sqrt{n}}}| + |q||e^{i\xi\frac{u}{\sqrt{n}}}|]
\end{aligned}$$

for any $\xi \in \mathbb{R}$. Because of the relationship between trigonometric functions and complex exponentials, $|e^{ia\xi\frac{u}{\sqrt{n}}}| \leq 1$ and clearly $|q| \leq 1$. Then we conclude that

$$\phi'''(\xi) = \frac{u^3}{n^{\frac{3}{2}}}h(\xi) \text{ in which } |h(\xi)| \leq 3.$$

The first three terms of the series have computed values $\psi(0)$, $\psi'(0)$, and $\psi''(0)$ to be

$$\begin{aligned}
\psi(0) &= 1 \\
\psi'(0) &= \frac{iu}{\sqrt{n}}(2q-1) \\
\psi''(0) &= \frac{-u^2}{n}.
\end{aligned}$$

Let's re-write $\log(\phi_n(t))$ in terms of the Taylor Series as

$$\frac{\log(1 + \frac{iu}{\sqrt{n}}(2q-1)t - \frac{u^2t^2}{2n} + \frac{u^3}{n^{\frac{3}{2}}}h(\xi))}{\frac{1}{n}}.$$

Then we multiply out terms to have

$$\frac{\log(1+\frac{2tiu}{\sqrt{n}(e^{\frac{u}{\sqrt{n}}}+1)}-\frac{tiu}{\sqrt{n}}-\frac{u^2t^2}{2n}+\frac{u^3}{n^{\frac{3}{2}}}h(\xi))}{\frac{1}{n}}.$$

Let $\alpha = \frac{\log(1+\frac{2tiu}{\sqrt{n}(e^{\frac{u}{\sqrt{n}}}+1)}-\frac{tiu}{\sqrt{n}}-\frac{u^2t^2}{2n}+\frac{u^3}{n^{\frac{3}{2}}}h(\xi))}{\frac{1}{n}}$. We want to study the behavior of α as $n \to \infty$. We observe that we have the indeterminate form $\frac{0}{0}$. From elementary calculus we can apply L'Hospital's rule. Taking the derivative of both the numerator and denominator we have

$$\beta = \frac{\frac{\frac{u^2ite^{\frac{u}{\sqrt{n}}}}{n^2(e^{\frac{u}{\sqrt{n}}}+1)^2}-\frac{uit}{n^{\frac{3}{2}}(e^{\frac{u}{\sqrt{n}}}+1)}+\frac{uit}{2n^{\frac{3}{2}}}+\frac{u^2t^2}{2n^2}-\frac{3u^3t^3}{4n^{\frac{5}{2}}}}{1+\frac{2tiu}{\sqrt{n}(e^{\frac{u}{\sqrt{n}}}+1)}-\frac{tiu}{\sqrt{n}}-\frac{u^2t^2}{2n}+\frac{u^3t^3}{2n^{\frac{3}{2}}}}}{\frac{-1}{n^2}}$$

Flipping the term in the denominator to the numerator and letting $n \to \infty$, what remains in the denominator will go to one. Then we are left looking at the expression

$$\kappa = -\frac{u^2ite^{\frac{u}{\sqrt{n}}}}{(e^{\frac{u}{\sqrt{n}}}+1)^2}+[\frac{uit\sqrt{n}}{e^{\frac{u}{\sqrt{n}}}+1}-\frac{uit\sqrt{n}}{2}]-\frac{u^2t^2}{2}+\frac{3u^3t^3}{4\sqrt{n}}.$$

The term to the left and the two terms to the right of the middle bracketed term converge to -$\frac{tiu^2}{4}$, $\frac{u^2t^2}{2}$, and 0 respectively. However, the middle terms yields another indeterminate form of $\frac{0}{0}$.Again, we apply L'Hospital's rule to the middle term to obtain

$$-\frac{tiu^2e^{\frac{u}{\sqrt{n}}}}{(e^{\frac{u}{\sqrt{n}}}+1)^2}.$$

Then as $n \to \infty$, this expression converges to $-\frac{tiu^2}{4}$. Adding together the limiting values of these four terms we get $-(\frac{ti}{2}+\frac{t^2}{2})u^2$. But this is the limit for the logarithm of the characteristic function of our random variable S_T. The limit of the logarithm is equal to the logarithm of the limit. This means that

$$\log \phi_n \to -(\frac{ti}{2}+\frac{t^2}{2})u^2 \Longrightarrow \phi_n = e^{\log \phi_n} \to e^{-(\frac{ti}{2}+\frac{t^2}{2})u^2} = \phi.$$

Now that we have shown that this limit exists, then by Levy's Continuity Theorem we can say that the sequence of CDF's does in fact converge to the normal $Z = N(-\frac{1}{2}u^2, u^2)$ since $e^{-}(\frac{ti}{2}+\frac{t^2}{2}u^2)$ is its characteristic function. ■

6 Explicit formula for C_0^n as $n \to \infty$

We previously found that

$$C_0^n = \sum_{i=0}^{n} \binom{n}{i} q^{n-i}(1-q)^i \max(S_0 e^{(n-i)u+id} - K, 0).$$

We simplify this equation to be

$$C_0^n = \sum_{i=0}^{n} \binom{n}{i} q^{n-i}(1-q)^i \max(S_0 e^{\frac{u}{\sqrt{n}}(n-2i)} - K, 0).$$

We notice that many terms in this summation would result in zero and that the only time that it would produce a zero is when

$$S_0 e^{\frac{u}{\sqrt{n}}(n-2i)} \leq K.$$

This means that the term would not be zero if

$$S_0 e^{\frac{u}{\sqrt{n}}(n-2i)} > K.$$

This is equivalent to

$$e^{\frac{u}{\sqrt{n}}(n-2i)} > \frac{K}{S_0}, \quad \text{since } S_0 > 0.$$

But,

$$\begin{aligned}
\log\left(e^{\frac{u}{\sqrt{n}}(n-2i)}\right) &> \log\left(\frac{K}{S_0}\right) \\
\iff \frac{u}{\sqrt{n}}(n-2i) &> \log\left(\frac{K}{S_0}\right) \\
\iff (n-2i) &> \frac{\sqrt{n}}{u}\log\left(\frac{K}{S_0}\right) \\
\iff -2i &> -n + \frac{\sqrt{n}}{u}\log\left(\frac{K}{S_0}\right) \\
\iff i &< \frac{n}{2} - \frac{\sqrt{n}}{2u}\log\left(\frac{K}{S_0}\right) \\
\iff i &< \frac{nu - \sqrt{n}\log\left(\frac{K}{S_0}\right)}{2u}.
\end{aligned}$$

This then tells us that

$$C_0^n = \sum_{i=0}^{\left\lfloor \frac{nu-\sqrt{n}\log\left(\frac{K}{S_0}\right)}{2u} \right\rfloor} \binom{n}{i} q^{n-i}(1-q)^i \left(S_0 e^{\frac{u}{\sqrt{n}}(n-2i)} - K\right).$$

Let $l_n = \left\lfloor \frac{nu-\sqrt{n}\log\left(\frac{K}{S_0}\right)}{2u} \right\rfloor = \frac{nu-\sqrt{n}\log\left(\frac{K}{S_0}\right)}{2u} - m_n$, where $0 \leq m_n < 1$. Now we can rewrite C_0^n as

$$C_0^n = \sum_{i=0}^{l_n} \binom{n}{i} q^{n-i}(1-q)^i \left(S_0 e^{\frac{u}{\sqrt{n}}(n-2i)}\right) - K \sum_{i=0}^{l_n} \binom{n}{i} q^{n-i}(1-q)^i.$$

Define $\alpha_n = \sum_{i=0}^{l_n} \binom{n}{i} q^{n-i}(1-q)^i \left(S_0 e^{\frac{u}{\sqrt{n}}(n-2i)}\right)$ and $\beta_n = -K \sum_{i=0}^{l_n} \binom{n}{i} q^{n-i}(1-q)^i$.

Notice that

$$\begin{aligned} F_n\left(\frac{u}{\sqrt{n}}(n-2i)\right) &= \sum_{j=i}^{n} \binom{n}{j} q^{n-j}(1-q)^j \\ &= \sum_{j=0}^{n} \binom{n}{j} q^{n-j}(1-q)^j - \sum_{j=0}^{i-1} \binom{n}{j} q^{n-j}(1-q)^j. \end{aligned}$$

If we let $l_n = i-1$, then

$$-\frac{\beta_n}{K} = \sum_{j=0}^{i-1} \binom{n}{j} q^{n-j}(1-q)^j.$$

We noticed a similarity to the CDF, so

$$F_n\left(\frac{u}{\sqrt{n}}\left(n-2\left(l_n+1\right)\right)\right) = \sum_{j=0}^{n} \binom{n}{j} q^{n-j}(1-q)^j + \frac{\beta_n}{K}.$$

Observe that $\sum_{j=0}^{n} \binom{n}{j} q^{n-j}(1-q)^j = 1$. Now we are left with

$$
\begin{aligned}
F_n\left(\frac{u}{\sqrt{n}}\left(n-2\left(l_n+1\right)\right)\right) &= 1+\frac{\beta_n}{K} \\
\frac{\beta_n}{K} &= F_n\left(\frac{u}{\sqrt{n}}\left(n-2\left(l_n+1\right)\right)\right)-1.
\end{aligned}
$$

We know that $F_n \to F$, so we need only to find $\lim_{n\to\infty}\left(\frac{u}{\sqrt{n}}\left(n-2\left(l_n+1\right)\right)\right)$

$$
\begin{aligned}
&= \lim_{n\to\infty}\left(\frac{u}{\sqrt{n}}\left(n-2l_n-2\right)\right) \\
&= \lim_{n\to\infty}\left(\frac{u}{\sqrt{n}}\left(n-2\left(\frac{nu-\sqrt{n}\log\left(\frac{K}{S_0}\right)}{2u}-m_n\right)-2\right)\right) \\
&= \lim_{n\to\infty}\left(\frac{u}{\sqrt{n}}\left(n-2\frac{nu-\sqrt{n}\log\left(\frac{K}{S_0}\right)}{2u}+2m_n-2\right)\right) \\
&= \lim_{n\to\infty}\left(\frac{u}{\sqrt{n}}\left(n-\frac{nu-\sqrt{n}\log\left(\frac{K}{S_0}\right)}{u}+2m_n-2\right)\right) \\
&= \lim_{n\to\infty}\frac{u}{\sqrt{n}}\left(\frac{un-un+\sqrt{n}\log\left(\frac{K}{S_0}\right)+2m_nu-2u}{u}\right) \\
&= \lim_{n\to\infty}\left(\frac{\sqrt{n}\log\left(\frac{K}{S_0}\right)+2m_nu-2u}{\sqrt{n}}\right) \\
&= \lim_{n\to\infty}\log\left(\frac{K}{S_0}\right)+\frac{2m_nu-2u}{\sqrt{n}} \\
&= \log\left(\frac{K}{S_0}\right). \qquad (1)
\end{aligned}
$$

Therefore we know that

$$
\begin{aligned}
\frac{1}{K}\lim_{n\to\infty}\beta_n &= -1+F\left(\log\left(\frac{K}{S_0}\right)\right), \quad \text{which implies} \\
\lim_{n\to\infty}\beta_n &= K\left(-1+F\left(\log\left(\frac{K}{S_0}\right)\right)\right).
\end{aligned}
$$

Now that we have found what $\lim_{n\to\infty}\beta_n$ is, we want to find $\lim_{n\to\infty}\alpha_n$. Recall that

$$\alpha_n = \left(\sum_{i=0}^{l_n}\binom{n}{i}q^{n-i}(1-q)^i\left(S_0 e^{\frac{u}{\sqrt{n}}(n-2i)}\right)\right)$$

Now remember that $\mathbb{E}S_n^n = \sum_{i=0}^{n}\binom{n}{i}q^{n-i}(1-q)^i S_0 e^{\frac{u}{\sqrt{n}}(n-2i)}$. Observe that

$$\begin{aligned}
\mathbb{E}S_n^n &= \sum_{i=0}^{n}\binom{n}{i}q^{n-i}(1-q)^i S_0 e^{\frac{u}{\sqrt{n}}(n-2i)} \\
&= \sum_{i=0}^{l_n}\binom{n}{i}q^{n-i}(1-q)^i S_0 e^{\frac{u}{\sqrt{n}}(n-2i)} \\
&+ \sum_{i=l_n+1}^{n}\binom{n}{i}q^{n-i}(1-q)^i S_0 e^{\frac{u}{\sqrt{n}}(n-2i)}. \qquad (2)
\end{aligned}$$

Notice that α_n looks similiar to the first sum in Equation 2. By substitution, Equation 2 can be rewritten as

$$\mathbb{E}S_n^n = \alpha_n + \sum_{i=l_n+1}^{n}\binom{n}{i}q^{n-i}(1-q)^i S_0 e^{\frac{u}{\sqrt{n}}(n-2i)}. \qquad (3)$$

Recall that $\mathbb{E}S_n^n = S_0$, so Equation 3, after solving for α_n, becomes

$$\begin{aligned}
\alpha_n &= S_0 - \sum_{i=l_n+1}^{n}\binom{n}{i}q^{n-i}(1-q)^i S_0 e^{\frac{u}{\sqrt{n}}(n-2i)} \\
&= S_0 - \sum_{i=l_n+1}^{n}\binom{n}{i}q^{n-i}(1-q)^i S_0 \left(e^{\frac{u}{\sqrt{n}}}\right)^{n-i}\left(e^{-\frac{u}{\sqrt{n}}}\right)^i \\
&= S_0 - \sum_{i=l_n+1}^{n}\binom{n}{i}S_0\left(e^{\frac{u}{\sqrt{n}}}q\right)^{n-i}\left(e^{\frac{u}{\sqrt{n}}}(1-q)\right))^i. \qquad (4)
\end{aligned}$$

Notice that $\left(e^{\frac{u}{\sqrt{n}}}q\right) = 1-q$ and $\left(e^{\frac{u}{\sqrt{n}}}(1-q)\right) = q$.

Let $e^{\frac{u}{\sqrt{n}}}q = \hat{q}$ and $\left(e^{\frac{u}{\sqrt{n}}}(1-q)\right) = 1-\hat{q}$. We can rewrite Equation 4 as

$$\alpha_n = S_0 - \sum_{i=l_n+1}^{n} \binom{n}{i} S_0 (\hat{q})^{n-i} (1-\hat{q}))^i \quad (5)$$

Similar to F_n, define $\hat{F_n}\left(\frac{u}{\sqrt{n}}(n-2i)\right) = \sum_{j=i}^{n} \binom{n}{j} \hat{q}^{n-j}(1-\hat{q})^j$. Recall that $l_n = i-1$, thus $i = l_n + 1$. Therefore

$$\hat{F_n}\left(\frac{u}{\sqrt{n}}(n-2i)\right) = \sum_{i=l_n+1}^{n} \binom{n}{i} \hat{q}^{n-i}(1-\hat{q})^i. \quad (6)$$

Recall Equation 5 as

$$\begin{aligned} \alpha_n &= S_0 - \sum_{i=l_n+1}^{n} \binom{n}{i} S_0 (\hat{q})^{n-i} (1-\hat{q}))^i \\ &= S_0 \left(1 - \sum_{i=l_n+1}^{n} \binom{n}{i} (\hat{q})^{n-i} (1-\hat{q}))^i \right). \quad (7) \end{aligned}$$

By combining Equation 6 and Equation 7, we have that

$$\alpha_n = S_0 \left(1 - \hat{F}\left(\frac{u}{\sqrt{n}}(n-2i)\right)\right).$$

Now we want to ensure that $\hat{F_n} \to \hat{F}$, where $\hat{F}$ is a CDF. Since the only difference between F_n and $\hat{F_n}$ is the $\hat{q}$, we check to see what happens with this term in the proof of Section 5. Notice that in the expression

$$\frac{\log(1 + \frac{iu}{\sqrt{n}}(2q-1)t - \frac{u^2t^2}{2n} + \frac{|u^3|t^3}{2n^{\frac{3}{2}}})}{\frac{1}{n}},$$

the only term with q is the t term, and this term is intimately related to the first moment or expectation. By substituting the q with $\hat{q}$, we get the following identities:

$$\begin{aligned} \frac{iu}{\sqrt{n}}(2\hat{q}-1)t &= \frac{iu}{\sqrt{n}}(2(1-q)-1)t \\ &= \frac{iu}{\sqrt{n}}(-2q+1)t \\ &= -\frac{iu}{\sqrt{n}}(2q-1)t. \end{aligned}$$

Then,

$$\lim_{n\to\infty} \frac{\log\left(1-\frac{iu}{\sqrt{n}}(2q-1)t-\frac{u^2t^2}{2n}+\frac{|u^3|t^3}{2n^{\frac{3}{2}}}\right)}{\frac{1}{n}},$$

we have that if we follow the proof of Theorem 1 with $F_n \to F$, the CDF's of $\hat{F}_n$ converge to the normal

$$Z = N(\frac{u^2}{2}, u^2).$$

We denote this CDF by $\hat{F}$.

Hence, we know from Equation 1 that $\lim_{n\to\infty}\left(\frac{u}{\sqrt{n}}(n-2i)\right) = \log\left(\frac{K}{S_0}\right)$. Thus,

$$\lim_{n\to\infty} \alpha_n = S_0\left(1-\hat{F}\left(\frac{K}{S_0}\right)\right).$$

Recall $C_0^n = \alpha_n + \beta_n$, therefore we know

$$\begin{aligned}
\lim_{n\to\infty} C_0^n &= \lim_{n\to\infty} \alpha_n + \beta_n \\
&= \lim_{n\to\infty} \alpha_n + \lim_{n\to\infty} \beta_n \\
&= S_0\left(1-\hat{F}\left(\log\left(\frac{K}{S_0}\right)\right)\right) \\
&- K\left(1-F\left(\log\left(\frac{K}{S_0}\right)\right)\right). \qquad (8)
\end{aligned}$$

We want to simplify our result in terms of standard normal distribution, so $F\left(\log\left(\frac{K}{S_0}\right)\right) = N\left(\frac{\log\left(\frac{K}{S_0}\right)-\mu}{\sigma}\right)$, in which N is the CDF of a normal $(0,1)$. For $\hat{F}$, $\mu = \frac{u^2}{2}$ and $\sigma = u$, and for F, $\mu = -\frac{u^2}{2}$ and $\sigma = u$. By Equation 8,

$$\begin{aligned}
\lim_{n\to\infty} C_0^n &= S_0\left(1-N\left(\frac{\log\left(\frac{K}{S_0}\right)-\frac{u^2}{2}}{u}\right)\right) \\
&- K\left(1-N\left(\frac{\log\left(\frac{K}{S_0}\right)+\frac{u^2}{2}}{u}\right)\right).
\end{aligned}$$

Observe that $1 - N(x) = N(-x)$ and $-\log\left(\frac{K}{S_0}\right) = \log\left(\frac{S_0}{K}\right)$, thus

$$
\begin{aligned}
\lim_{n\to\infty} C_0^n \equiv C_0^\infty & \\
&= S_0\left(N\left(\frac{\log\left(\frac{S_0}{K}\right)+\frac{u^2}{2}}{u}\right)\right) - K\left(N\left(\frac{\log\left(\frac{S_0}{K}\right)-\frac{u^2}{2}}{u}\right)\right).
\end{aligned}
$$

References

[1] J. Jacod, and P. Protter, *Probability Essentials*, Springer-Verlag Berlin Heidelberg, Germany, 2004, pp. 167, 181-183.