On the properties and structure of 4×4 Sudoku tables, with 9×9 extensions

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Background of Sudoku

A Sudoku puzzle is a table made up of 9 rows, 9 columns and 9, 3×3 boxes. The puzzle starts with given numbers in various positions and the player's goal is to complete the table such that each row, column and box contains every number in the set $\{1, \ldots, 9\}$ exactly once. Each puzzle has a unique solution, yet the clues given are not enough to make the puzzle trivial.

		9	1		7	6		3
	6			3	4			7
					1	4		2
	3	4	8		6	5	7	
2		7	3					
5			7	1			8	
7		1	5		9	3		

These puzzles resemble Latin squares, with the further restriction that each number must appear once in each box. The number of 9×9 Sudoku tables is approximately 6.671×10^{21} [1]. The number of "classes" of Sudoku tables is 5,472,730,538 [2]. Since the 9×9 Sudoku tables yield such large numbers, we decided to work with 4×4 Sudoku tables and puzzles.

Shidoku tables A Shidoku table is a Latin square of size 4 in which each 2×2 box has each element of the set $\{1, \ldots, 4\}$ appearing exactly once.

We note that the results we obtained arose through a learning by discovery project. We did not consult the literature until after we concluded our explorations. Thus, in some instances, we derived results that have appeared in print or are awaiting publication [3].

How many Shidoku tables are there?

Theorem 1 *There are 288 distinct Shidoku tables.*

Lemma 1 *Let U and L be upper and lower diagonal quadrants of a Shidoku table. Suppose the entries of U are known. Then, each row of U shares exactly one element with each column of L. Similarly, each column of U shares exactly one element with each row of L.*

There are $4 \times 3 \times 2 \times 1 = 4! = 24$ possible rearrangements for the 2×2 boxes of a Shidoku table.

a_{11}	a_{12}		
a_{21}	a_{22}		
		a_{33}	a_{34}
		a_{43}	a_{44}

- If we fix the upper left hand quadrant of a table, by the lemma above, we can choose only 12 of the 24 2×2 boxes to put in the lower right hand quadrant of the table.
- Observe that if we fix the upper left hand quadrant and the lower right hand quadrant, then there is only one way to fill the remaining entries of the 4×4 partially filled table.
- So, we have 24 possible upper left hand quadrants. For each of these there correspond 12 possible lower right hand quadrants, and each of these pairs will result in a Shidoku table.
- Thus, we arrive at a total of total of $24 \times 12 = 288$ Shidoku tables.

Can these tables be divided into classes?

We want to explore "equivalent" Shidoku tables by considering valid transformations.

- We call two tables equivalent if we can obtain one from another by a series of valid transformations.
- These valid transformations are relabeling, switching rows and columns and taking the transpose.
- We consider a table ordered if it is in the following form:

a	b	c	d
c	d		
b			
d			

.

Through ordering and relabeling we find that there are two equivalence classes of Shidoku tables, which are

a	b	c	d
c	d	a	b
b	a	d	c
d	c	b	a

A

a	b	c	d
c	d	a	b
b	c	d	a
d	a	b	c

B

Through this ordering and relabeling process we also confirm that there are 288 unique Shidoku tables.

How many Shidoku tables come from Cayley tables of groups of order 4?

- We consider the groups that have four elements: $\mathbb{Z}_4$ and the Klein-4 group.
- An example of the Klein-4 group is the set of natural numbers relatively prime to 8, $\{1, 3, 5, 7\}$, under the operation of multiplication modulo 8.
- An example of $\mathbb{Z}_4$ is the set of integers $\{0, 1, 2, 3\}$ under the operation of addition modulo 4.
- Cayley tables of the Klein-4 group and of $\mathbb{Z}_4$ are shown below, respectively.

*	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

*	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Theorem 2 *If a Cayley table of the Klein-4 group has Shidoku structure, then after ordering and relabeling it is an element of class A.*

Theorem 3 *If a Cayley table of the $\mathbb{Z}_4$ group has Shidoku structure, then after ordering and relabeling it is an element of class B.*

Therefore, all Shidoku tables come from Cayley tables of Kline 4 or $\mathbb{Z}_4$ by ordering and relabeling.

What is the least number of clues that can uniquely determine a Shidoku puzzle?

Theorem 4 *In order for a Shidoku puzzle to yield a unique table as its solution, it must contain at least 4 clues.*

- Observe the puzzle below on the left and its possible solutions on the right produced by "ambiguous pairs" of entries:

a	b	c	d
c	d	a	b
b		d	
d		b	

$\Rightarrow$

a	b	**c**	d
c	d	**a**	b
b	**a**	d	**c**
d	**c**	b	**a**

a	b	c	d
c	d	a	b
b	c	d	a
d	a	b	c

- The solutions are the representative tables of both classes A and B.
- Table B has only four ambiguous pairs.
- In table B, at least one element per pair must be determined to get a unique table.
- Only 3 clues leaves an ambiguous pair undetermined in table B.
- In table A, we have 8 ambiguous pairs.
- Using 3 clues will leave two ambiguous pairs undetermined.
- Nonetheless, 4 carefully placed clues will determine each pair since their elements overlap.
- Finally, note that the set of clues must have at least 3 distinct entries.

Extension to 9x9 Sudoku Tables and Conclusion

- We get an upper statistical estimate of about $5.32 * 10^8$ Sudoku tables formed from Cayley tables of groups of order 9.
- Using arguments analogous to those in the 4x4 case, we show that there are $6^4 * 6^4 * 2 * 9!$ symmetric transformation of these tables, yielding fewer than $6.48 * 10^{20}$ tables from groups of order 9.
- There are over $6.22 * 10^{21}$ Sudoku tables.
- These findings indicate that although all Shidoku tables come from groups of order 4, the majority of Sudoku tables do not come from groups of order 9.

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References

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[3] Taalman, L., Taking Sudoku Seriously, *Math. Horizons* (to appear).