Liquidity Risk and Trade Impact

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The CJP model - Cetin, Jarrow, Protter (2004)

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- $xS(t,x) = S_t x + M_t x^2$. Hence the marginal cost is $S_t + 2M_t x$.
- Self-financing strategies (X, Y) satisfy

$$Y_T = Y_0 + \int_0^T X_{u-} dS_u - \int_0^T M_u d[X]_u.$$

X_t denotes the number of shares held at time t and Y_t the money in the bank account (0 interest).

Figure: Typical order book density for linear model.

Figure: Order book is partly used up.

Figure: Price Impact at time $t+$.

Mathematical Framework

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- We denote by S_t the marginal price (or unaffected price) that would have been observed if not trades had been executed until time t $(X_s = 0,\ 0 \leq s \leq t)$.

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- We denote by S_t the marginal price (or unaffected price) that would have been observed if not trades had been executed until time t $(X_s = 0,\ 0 \leq s \leq t)$.
- **Second type of resiliency : damping.** In the long run, the effect of past trades on prices decreases : price S_t^0 converges to the unaffected price S_t.

Figure: Typical sample path

Mathematical Framework

- We define the price after impact by

$$S^0_{t+} = S_t + 2\lambda \int_0^t e^{-\kappa(t-u)} M_u dX_u + 2\lambda \int_0^t e^{-\kappa(t-u)} d[M, X]_u.$$

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- Let $\sigma_n : 0 = \tau^n_0 \leq \tau^n_1 \leq \ldots \leq \tau^n_{k_n} = t$ be a sequence of random partitions tending to the identity and $\Delta^n_k X = X_{\tau^n_k} - X_{\tau^n_{k-1}}$. A pair $(X_t, Y_t)_{t \geq 0}$ is a self-financing trading strategy (s.f.t.s) if X is a cadlag process and Y is an optional process satisfying

$$Y_t = Y_0 - \lim_{n\to\infty} \sum_{k=1}^{k_n} \Delta^n_k X S^0(\tau^n_k, \Delta^n_k X).$$

We will always define trading strategies with $X_{0-} = Y_{0-} = 0$.

Self-financing Strategies

Theorem 1

Let X be a FV cadlag process and Y an optional process. Define

$$
\begin{aligned}
I_t &= \lambda \int_0^t X_{u-}^2 \, dm_u \\
L_t &= \int_0^t K(t-u) M_u d[X,X]_u + (1-\lambda) M_t X_t^2
\end{aligned}
$$

with $K(t) = 1 - \lambda e^{-\kappa t}$ and $m_t = e^{-\kappa t} M_t$. If $(X_t, Y_t)_{t \geq 0}$ is a self-financing trading strategy then

$$
Y_T + X_T S^0_{T+}(-X_T) = Y_0 + X_0 S_0(X_0) + \int_0^T X_{u-} dS_u - L_T - I_T
$$

- The case $\lambda = 0$ corresponds to the CJP model. (Full resiliency)

No Arbitrage and Equivalent Martingale Measures

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- Profit $= \int_0^T X_{u-} dS_u - \lambda \int_0^t X_{u-}^2 dm_u -$ Liquidity Costs.

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$$\begin{aligned} dv_t &= \tilde{a}(v_s, s)ds + b_1(v_s, s)dW_{1,s} + b_2(v_s, s)dW_{2,s} \\ dm_t &= \eta(m_s, s)ds + \xi_1(m_s, s)dW_{1,s} + \xi_2(m_s, s)dW_{2,s} \end{aligned}$$

for all $0 \leq t \leq T$ in which $\widetilde{a}, b, \xi, \eta$... are Lipschitz functions which ensures the existence of such processes. $m_t = \exp(-\kappa(T-t))M_t$. v_t is a component of the stochastic volatility.

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- Furthermore we will assume that the matrix

$$\Sigma(m_t, v_t, t) = \begin{pmatrix} b_1(v_t, t) & b_2(v_t, t) \\ \xi_1(m_t, t) & \xi_2(m_t, t) \end{pmatrix}$$

  is invertible for all $0 \le t \le T$.

- Recall that S_t is the stock price resulting in the actions of all investors in the market except me. The aggregated effect of trading done by investor i on the stock price is $2\lambda \int_0^t m_u dX_u^i + 2\lambda [m, X^i]_t$.

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- We can expect the stock price to be affected by sum over all investors in the market (except me) and some other risk source : $S_T = S_0 + \sum_i 2\lambda \int_0^T m_u dX_u^i + 2\lambda \sum_i [m, X^i]_T + \int_0^T v_u S_u dW_u$.

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- If we assume $\sum_i dX_t^i$ is of the form $S_t dW_t$, the stock process is then given by

$$dS_t = \mu_t S_t dt + \sum_{i=1}^{3} \sigma_i \sigma_t S_t dW_{i,t}$$

for all $0 \leq t \leq T$ in which $\sigma_t = m_t + v_t$.

The Volatility Swaps

- We add two volatility swaps, denoted $G_{i,t}$ for $i = 1, 2$. To ensure no arbitrage, we assume the existence of an equivalent probability measure $\mathcal{Q}$ such that S is martingale, m is submartingale and

$$G_{i,t} = \mathbf{E}_{\mathcal{Q}}\left(\sigma_{T_i} \middle| \mathcal{F}_t\right)$$

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- S.f.t.s now satisfy

$$\begin{aligned} Y_T &= Y_0 + \int_0^T X_{u-} dS_u - \lambda \int_0^T X_{u-}^2 dm_u - \int_0^T K(T-u) M_u d[X]_u \\ &\quad + \sum_i \int_0^T \chi_{i,u-} dG_{i,u} - \sum_i \int_0^T K_i(T-u) N_i d[\chi_i]_u. \end{aligned}$$

  Here N_i and K_i refer to the liquidity constraints of G_i.

Girsanov's Theorem

- By Girsanov's theorem, there exists a predictable process θ such that under $\mathcal{Q}$

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- This essentially means that

$$\begin{aligned}(\theta_{1,t}\sigma_1 + \theta_{2,t}\sigma_2 + \theta_{3,t}\sigma_3)\sigma_t &= \mu_t \quad \text{and} \\ (\theta_{1,t}\xi_1 + \theta_{2,t}\xi_2) &\leq \eta(m_t, t).\end{aligned}$$

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- Let $\zeta_t = \eta(m_t, t) - (\theta_{1,t}\xi_1(m_t, t) + \theta_{2,t}\xi_2(m_t, t))$ and $a_t = \tilde{a}(v_t, t) - (\theta_{1,t}b_1(v_t, t) + \theta_{2,t}b_2(v_t, t))$. Then

$$\begin{aligned}dv_t &= a_t dt + b_1(v_t, t)dB_{1,t} + b_2(v_t, t)dB_{2,t} \\ dm_t &= \zeta_t dt + \xi_1(m_t, t)dB_{1,t} + \xi_2(m_t, t)dB_{2,t} \\ dS_t &= \sum_i \sigma_i \sigma_t S_t dB_{i,t}.\end{aligned}$$

Approximate Completeness and S.f.t.s.

- Recall the definition of self-financing :

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Lemma 1

Fix t and let $H_T \in \mathcal{L}^\infty$. Suppose there exist predictable processes X and χ with $H_T = c + \int_t^T X_u dS_u + \sum_{i=1,2} \int_t^T \chi_{i,u} dG_{i,u} - \lambda \int_t^T X_u^2 dm_u$ for some $c \in \mathbb{R}$. Then there exists a sequence of s.f.t.s. (X^n, χ^n, Y^n) with X^n bounded, continuous and of finite variation such that $X_t^n = 0$, $X_T^n = 0, \chi_t^n = 0, \chi_T^n = 0$ and $Y_t^n = \mathbf{E}_{\mathcal{Q}} \left(H_T + \lambda \int_t^T (X_{u-}^n)^2 m_u \zeta_u du \middle| \mathcal{F}_t \right)$ for all n and $Y_T^n \to H_T$ in $\mathcal{L}^2(d\mathcal{Q})$.

Quadratic Growth BSDEs

- For a given payoff H_T, the replication problem boils down to finding processes X, χ and a constant c that satisfy:

$$H_T = c + \int_t^T X_u dS_u + \sum_{i=1,2} \int_t^T \chi_{i,u} dG_{i,u} - \lambda \int_t^T X_u^2 dm_u.$$

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- The existence of such a process is given by the existence of a solution to quadratic growth BSDEs :

Theorem 3

Let $0 \leq t_0 \leq T$ and $T_1 \neq T_2$. Suppose $\zeta_t = \zeta m_t$ and $a_t = av_t$ for some constants $\zeta \neq a$. For $H \in \mathcal{L}^\infty(\mathcal{F}_T)$, there exists a unique solution $(X_t, \chi_t, Y_t)_{t_0 \leq t \leq T}$ to the following BSDE

$$Y_t = H - \int_t^T X_s dS_s + \lambda \int_t^T X_s^2 dm_s - \sum_i \int_t^T \chi_{i,s} dG_{i,s} \qquad (1)$$

for $t_0 \leq t \leq T$.

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- Let $(X^\epsilon, \chi^\epsilon, Y^\epsilon)$ be the solution of BSDE with terminal condition ϵH.
- Let $H_t(\epsilon)$ be the replicating price per share of ϵ shares of the claim H starting at time t, i.e. $\frac{Y_t^\epsilon}{\epsilon}$.

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- Let $H_t(\epsilon)$ be the replicating price per share of ϵ shares of the claim H starting at time t, i.e. $\frac{Y_t^\epsilon}{\epsilon}$.
- Denote $H_t(0) = \lim_{\epsilon \to 0} H_t(\epsilon) = \lim_{\epsilon \to 0} \frac{Y_t^\epsilon}{\epsilon}$ and $H_t'(0)$ the derivative at zero (we will see that it exists).

- The BSDE for ϵH can be written in the form:

$$Y_t^\epsilon = \epsilon H + \int_t^T \zeta m_s \lambda (X_s^\epsilon)^2 ds + \sum_i \int_t^T \Big(-X_s^\epsilon \sigma_i \sigma_s + \lambda (X_s^\epsilon)^2 \xi(m_s, s) - \sum_j \chi_{j,s}^\epsilon e^{-a(s-T_j)} b_j(v_s, s) \Big) dB_{i,s}$$

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Theorem 4

Let $H \in \mathcal{L}^\infty(\mathcal{F}_T)$. Then, if (X, χ, Y) denotes the solution of the BSDE with $\zeta = 0$ and $\epsilon = 1$, we have that $H_t(0) = Y_t = \mathbf{E}_{\mathcal{Q}}(H|\mathcal{F}_t)$ and $\frac{1}{\epsilon} X^\epsilon \to X$ in $\mathcal{L}^2(dQ \times dt)$ as $\epsilon \to 0$.

- In the results above, H cannot depend on X. We don't know the existence of a solution to the equation

$$\epsilon h(S_T - 2\lambda \int_0^T \widetilde{X}_s^\epsilon dm_s) = \widetilde{Y}_t^\epsilon - \int_t^T \widetilde{X}_s^\epsilon dS_s + \lambda \int_t^T (\widetilde{X}_s^\epsilon)^2 dm_s - \sum_i \int_t^T \widetilde{\chi}_{i,s}^\epsilon dG_{i,s}.$$

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Theorem 5

If h is Lipschitz continuous and bounded then

$$\sqrt{\mathbf{E}_{\mathcal{Q}} \left| \epsilon h(S_T - 2\lambda \int_0^T X_s^\epsilon dm_s) - \epsilon h(S_T - 2\lambda \int_0^T \epsilon X_s dm_s) \right|^2} = O(\epsilon^{2.5}).$$

Furthermore, if h is twice differentiable and its second derivative is bounded, then

$$H_t'(0) = \mathbf{E}_{\mathcal{Q}} \left(\int_t^T \zeta m_s X_s^2 ds \middle| \mathcal{F}_t \right) - 2\lambda \mathbf{E}_{\mathcal{Q}} \left(h'(S_T) (\int_t^T X_s dm_s) \middle| \mathcal{F}_t \right).$$

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